Basic concepts in astrophysics

1.1. Big Bang nucleosynthesis

first approximation: the matter in the universe consists of hydrogene and helium

the bulk of helium was produced by nuclear reactions in the first few minutes of the universe

$\Downarrow$

primordial nucleosynthesis

A brief history of the universe

- expanding → age ≈ 14 Billion years ($13.7 \cdot 10^9 a$)
- the whole space is filled with a thermal radiation field

↳ 3K Black body radiation: isotrop (Penzias, Wilson 1965)

Timeline:

Nanoseconds after Big Bang: Plasma of free fundamental particles (e^-, Quarks, Antiquarks, e^+)

Antimatter annihilated

↳ if $T < 10^{14}$ K ⇒ Baryon-synthesis

free Quarks and Antiquarks disappear

Milliseconds to one second: Plasma of free protons, neutrons and electrons, embedded in hot Photons

Neutrino field (not detected)

↳ if $T \sim 10^{10}$ K decoupling of Neutrinos

primordial neutrino field

↳ $T(\text{now}) \approx 2$ K

≈ 100 seconds: Primordial nucleosynthesis

first element synthese

75 % hydrogene
25 % helium
(mass procente)

was determined by the ratio of free neutrons to protons in the Temperature range ⇒ nucleosynthesis after 300 000 years: Temperature ~ 4000 K ⇒ forms stable Atoms

Background radiation field

free electrons disappear ↗ neutral H, " He

Photons leave the matter

↳ the universe became transparent

⇓

isotrop background radiation

→ adiabatic cooling $\underbrace{4000\,K \text{ to } 3\,K}_{13 \text{ Billion years}}$ now

↳ matter is clumping in Stars, Gas } $\underbrace{\text{Galaxies}}_{\text{Clusters of galaxies}}$ (begin ca 1 Million years after big bang)

<u>The synthesis of helium</u>

conditions: $T \approx 10^{10}$ to 10^9 K

<u>in normal circumstances</u> → Neutron → β-decay with a mean life time of about 15 minutes

$$n \rightarrow p + e^- + \bar{\nu}_e$$

but at <u>high temperature and density</u> → n can transformed to p and p can transformed to n in collisions with thermal leptons $(\nu, \bar{\nu}, e^-, e^+)$ } equilibrium

$$\nu_e + n \rightleftarrows e^- + p \quad \text{and} \quad \bar{\nu}_e + p \rightleftarrows e^+ + n$$

→ temperature decreased → neutrons were outnumbered by protons

$$\text{Ratio:} \quad \frac{N_n}{N_p} = \exp\left[-\frac{\Delta m c^2}{kT}\right] \qquad \Delta m = |m_p - m_n| \approx 1.3\,MeV/c^2$$

the Boltzmann-factor says: the n to p ratio decreased rapidly by cooling

↳ neutrino-reactions became less frequent

↳ neutrons were free and decay with a lifetime of ca. 15 minutes

↙ but $T < 10^9$ K

deuterium

<u>fusion reaction</u>: $n + p \rightarrow d + \gamma$

<u>photodisintegration</u>: $\gamma + d \rightarrow n + p$

- Deuterium-capture n, p → Tritium or Helium-3
- Tritium, ^3He-capture n, p → $\underbrace{\text{Helium 4}}_{\text{very stable}}$

moreover → formation of small amounts of Lithium

} primordial Helium and Lithium

<u>After three minutes</u>: 75 mass % hydrogen
25 mass % helium + a little Lithium + D
no free neutrons

Estimate n/p-ratio before big bang nucleosynthesis → 1/7

→ <u>of one neutron came 7 protons</u>

1.2. Gravitational contraction

Gravity leads to the compression of matter

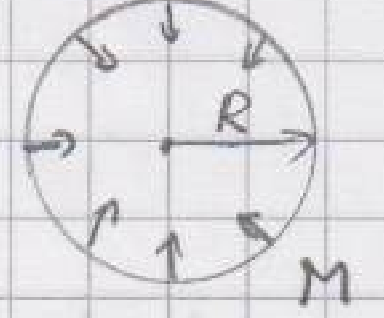
↳ cause for the Star-formation

<u>Theory of gravitational collapse</u>: Simple model: spherical system of mass M and radius R

mass accelaration:

$\varrho(r)$

$P(r)$



$$dV = \frac{4}{3}\pi\left((r+dr)^3 - r^3\right)$$

$$= 4\pi r^2 dr$$

$$dm = \varrho \cdot dV = 4\pi \varrho r^2 dr \qquad m(r) = \int_0^r \varrho(r')\, 4\pi r'^2 \, dr'$$

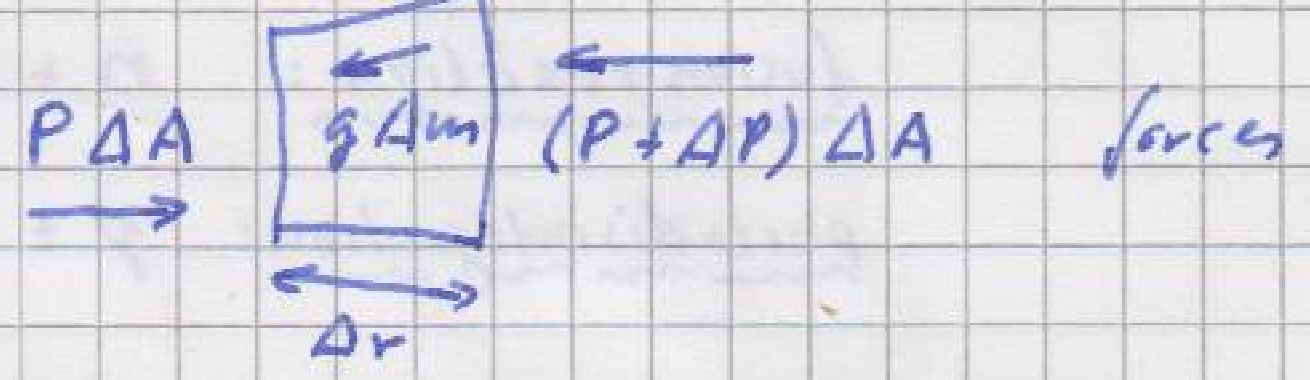
mass enclosed by a spherical shell

$$g(r) = \frac{Gm(r)}{r^2} \qquad \text{because} \quad F_g = \frac{GmM}{r^2} = m\frac{d^2r}{dt^2}$$

must be in equilibrium with a force arising from the pressure gradient

by r → volume element located between r and r + Δr

$$\Delta V = \Delta r \cdot \underbrace{\Delta A}_{\text{area}}$$



Equilibrium condition:

$$\left[P(r) + \frac{dP}{dr}\Delta r - P(r)\right]\Delta A = \underbrace{\frac{dP}{dr}}_{\text{pressure gradient}}\Delta r\, \Delta A$$

$$\Delta M = \rho(r)\,\Delta r\,\Delta A$$

$$\boxed{-\frac{d^2r}{dt^2} = g(r) + \frac{1}{\rho(r)}\frac{dP}{dr}}$$

if $|d^2r/dt^2| = 0$

then hydrostatical equilibrium

the pressure is increased toward the centre

Free Fall

What is happen when there is no pressure gradient to oppose gravitional collapse?

→ every mass element move towards the centre with a accelaration

$$g(r) = Gm(r)/r^2$$

1.) inward velocity $\qquad \frac{1}{2}\left[\frac{dr}{dt}\right]^2 = \frac{Gm_0}{r} - \frac{Gm_0}{r_0}$

(energy conversation)

$$\frac{M}{2}v^2 = \Delta E_{pot}$$

Time for free fall:

$$t_{ff} = \int_{r_0}^{0} \frac{dt}{dr}\, dr = -\int_{r_0}^{0} \left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2} dr$$

and with $x = r/r_0$

$$t_{ff} = \left[\frac{r_0^3}{2Gm_0}\right]^{1/2} \underbrace{\int_0^1 \left[\frac{x}{1-x}\right]^{1/2} dx}_{= \pi/2} = \left[\frac{\pi^2 r_0^3}{8Gm}\right]^{1/2}$$

$x = \sin^2 \Theta$

The free fall time for a shell of radius r enclosing mass m_0 is only determined by the average density of the enclosed matter

In the absence of an internal pressure gradient, a sphere with an initial, uniform density ϱ will collapse as a whole in a time

$$\boxed{t_{ff} = \left[\frac{3\pi}{32 G\varrho}\right]^{1/2}}$$

because $r_0^3 = 3m / 4\pi\varrho$

Example: - Sun $\sim$ 30 min

- Galaxies (with a density of 10^{-20} kg m^{-3}) $\sim 2.1 \times 10^7$ a

The collapse of a gravitational system leads to a energy dissipation that can usually stop the contraction

↳ increase temperature → increase pressure

if matter opaque...

oppose further collapse

⇓

state of hydrostatic equilibrium

Hydrostatic equilibrium

pressure gradient: $\underbrace{\frac{dP}{dr} = -\frac{G m(r) \varrho(r)}{r^2}}$

The whole system is in equilibrium if this equation valid at all radii r

$$\underbrace{\int_0^R 4\pi r^3 \frac{dP}{dr}\, dr = -\int_0^R \frac{G m(r) \varrho(r)\, 4\pi r^2}{r}\, dr}$$

$\varrho(r) = m(r) / V(r) = 3m(r) / 4\pi \varrho(r)$

$$\hookrightarrow -\int_{m=0}^{m=M} \frac{G m(r)}{r}\, dm = E_{GR}$$

gravitational potential energy

left hand side integrated by parts: $\int u v' dx = uv - \int u' v\, dx$

$u = 4\pi r^3 \quad u' = 12\pi r^2 \quad v = P(r)$

$$4\pi \int_0^R r^3 \frac{dP}{dr}\, dr = \underbrace{\Big[P(r)\, 4\pi r^3 \Big]_0^R}_{P(R) = 0 \text{ („Star surface")}} - 3\int_0^R P(r)\, 4\pi r^2\, dr$$

$$= -3\int_0^R P(r)\, 4\pi r^2\, dr = -3 \underbrace{\langle P \rangle}_{\text{volume-averaged pressure}} V$$

$= -4\pi P(r) R^3$ and $V = \frac{4}{3}\pi R^3$

$$\boxed{\langle P \rangle = -\frac{1}{3} \frac{E_{GR}}{V}}$$

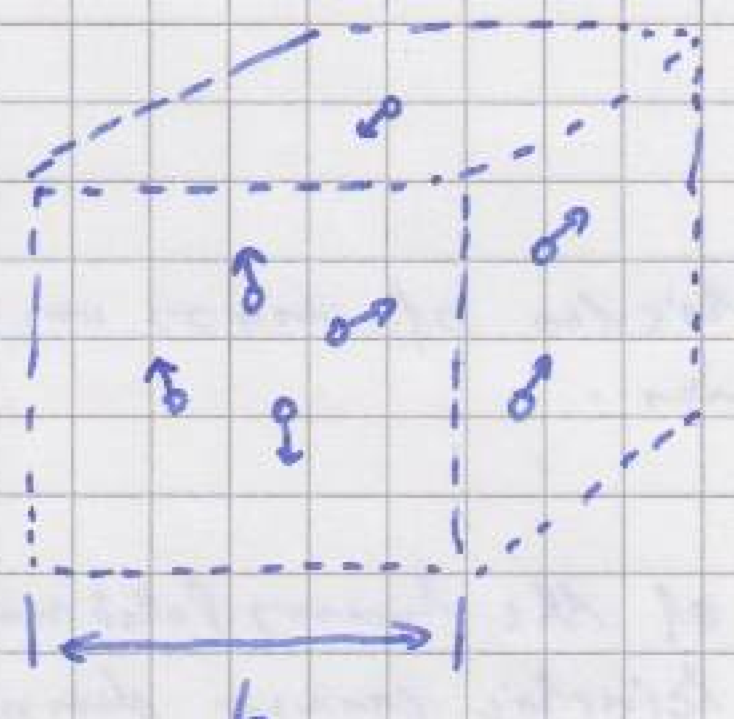
Virial theorem

What is the physical origin of the pressure?

$\Rightarrow$ gas of N particles in a box with $V = L^3$

a) velocity of a individual gas particle: $\vec{v} = (v_x, v_y, v_z)$

momentum " " " " " " : $\vec{p} = (p_x, p_y, p_z)$



total reflection on the box wall

Z-Axis:

rate at which particles strikes one of the sides perpendicular to the z-axis:

$\underbrace{v_z / 2L}_{\text{"avg"}}$ $\Rightarrow$ "pressure" $P = \frac{N}{L^3} \langle p_z v_z \rangle$

$\Rightarrow$ imparts a momentum of $2p_z$

But we have 6 sides: $\langle p_z v_z \rangle = \langle p_y v_y \rangle = \langle p_x v_x \rangle = \frac{\langle \vec{p}\vec{v} \rangle}{3}$

This gives the pressure on each side of the box:

$$\boxed{P = \frac{n}{3} \langle \vec{p} \cdot \vec{v} \rangle}$$

n = number of particles per unit volume

without interaction between particles: Ideal gas

This is valid for a classical gas. What is the behavior of a gas of relativistic particles ($v \approx c$)?

The general relation between energy and momentum of a particle of mass m is:

$$\boxed{\epsilon_p^2 = p^2c^2 + m^2c^4}$$

$E_K = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$ $\quad \epsilon_p = mc^2 + E_K$ $\quad v = \frac{pc^2}{\epsilon_p}$

non-relativistic limit: $p \ll mc$

$$\epsilon_p = mc^2 + \frac{p^2}{2m} \qquad v = p/m$$

ultra-relativistic limit: $p \gg mc$

$$\epsilon_p = pc \quad \text{and} \quad v = c$$

- For a gas of non-relativistic particles of mass m, $\vec{p}\cdot\vec{v} = mv^2$ and the pressure becomes:

$$P = \frac{2}{3} n \underbrace{\left\langle \frac{1}{2} m v^2 \right\rangle}_{\text{average kin. energy per particle}} = \frac{2}{3} \text{ of the translational kinetic energy density}$$

- For a gas of ultra-relativistic particles $\vec{p}\cdot\vec{v} = pc$ and the pressure becomes

$$P = \frac{1}{3} n \langle pc \rangle = \frac{1}{3} \text{ of the translational kinetic energy density}$$

Equilibrium of a gas of non-relativistic particles

Gas in a volume V held together by gravity:

$$N = \frac{\text{all particles}}{\text{entire volume}} \approx \frac{1}{V} \qquad \langle P \rangle = \frac{2}{3} \frac{E_{KE}}{V} \quad \Big\} \; E_{KE} = \underbrace{\sum_{i=1}^{N} E_{K_i}}_{\text{translation kinetic energy}}$$

and because: $\langle P \rangle = -\frac{1}{3} \frac{E_{GR}}{V}$

$$\boxed{2E_{KE} + E_{GR} = 0} \quad \text{virial theorem}$$

If the particles have no internal excited degrees of freedom, the total energy of the gas is the sum of the kinetic and gravitational energies of the particles, e.d. $E_{TOT} = E_{KE} + E_{GR}$

This implies that this total energy can be expressed in terms of either the kinetic energy or the gravitational energy of the particles:

$$\boxed{E_{TOT} = - E_{KE} \qquad E_{TOT} = \frac{1}{2} E_{GR}}$$

Very important!

This expressions describe the implication of the virial theorem for a system of selfgraviting, non relativistic particles in hydrostatic equilibrium

- E_{TOT} is the binding Energy of the system

=> a tightly bound gas cloud is hotter

=> compression of a gas cloud → increase temperature

If the system evolve slowly (always near the hydrostatical equilibrium)

↳ the change in the gravitational and kinetic energies are simply related to the change of the total energy:

Example: 1% decrease in the total Energie would be accompanied by a 2% decrease in the gravitational energy and 1% increase in the $\underbrace{\text{kinetic energy}}_{\text{increase temperature}}$

For a contraction close to hydrostatic equilibrium, half the gravitational energy released is lost from the surface by radiation and the other half is dissipated as heat.

Stability criteria

If the energy loss from the surface can be supplied by the release of nuclear energy by thermonuclear fusion, the total energy $E_{KE} + E_{GR}$ remains constant and there is no need for the cloud to contract.

↳ Example: main sequence stars

Equilibrium of a gas of ultra-relativistic particles

$P = \frac{1}{3} n \langle pc \rangle$ and $\langle P \rangle = -\frac{1}{3} \frac{E_{GR}}{V}$

$$\boxed{E_{KE} + E_{GR} = 0}$$

Hydrostatic equilibrium is possible only if the binding energy is zero.

→ System befindet sich im Schnittpunkt zwischen „gebunden" und „ungebunden"!

→ ultra-relativistic limit: binding energy decreased and the system is easily disrupted

- photons; by stars near at the Eddington luminosity
- degenerate electrons;

} instabilities

Equilibrium and the adiabatic index

adiabatic index γ: used to describe the relation between the pressure P and the volume V of a (ideal) gas during an adiabatic compression or expansion

Adiabatische Zustandsänderung

Kein Wärmeaustausch mit Umgebung

$$\boxed{P \cdot V^{\gamma} = \text{const}}$$

$\gamma P dV + V dP = 0 \quad | + PdV - PdV$

$\gamma \frac{dV}{V} + \frac{dP}{P} = 0 \quad \Rightarrow \quad d(PV) = PdV + VdP = -(\gamma - 1) P dV$

Introduction internal energy E_{IN}

change of internal energy caused by change of volume:

$dE_{IN} = -PdV$

$dE_{IN} = \frac{1}{\gamma - 1} d(PV)$

and $d(PV) = (\gamma - 1)\, dE_{IN}$

$$\boxed{E_{IN} = \frac{1}{(\gamma-1)} PV}$$

We consider a self-gravitating system of a ideal gas with adiabatic exponent γ, which is in hydrostatic equilibrium:

$$\langle P \rangle = (\gamma - 1)\frac{E_{IN}}{V} = -\frac{1}{3}\frac{E_{GR}}{V}$$

umstellen: $$\boxed{3(\gamma-1)E_{IN} + E_{GR} = 0}$$

$\Rightarrow$ gas of particles with no excited internal degrees of freedom

$\hookrightarrow$ $E_{IN} = E_{KE}$ (internal energy = kinetic energy)

$\gamma = 5/3$ non relativistic

$\gamma = 4/3$ ultra relativistic

hydrostatic equilibrium

$$E_{TOT} = E_{IN} + E_{GR} = -(3\gamma - 4)E_{IN}$$

$E_{IN} + E_{GR} - E_{TOT} = 0$

$E_{GR} = -3(\gamma-1)E_{IN}$

A gas is bound if $\gamma > 4/3$

Implications:

- the binding energy is small near $\gamma \approx 4/3$

  $\hookrightarrow$ a small change in E_{TOT} $\rightarrow$ is accompanied by a much larger changes in E_{IN} and E_{GR}

- Instability is expected whenever γ is reduced towards $4/3$.

1.3. Star formation

- the most stars are formed in clusters

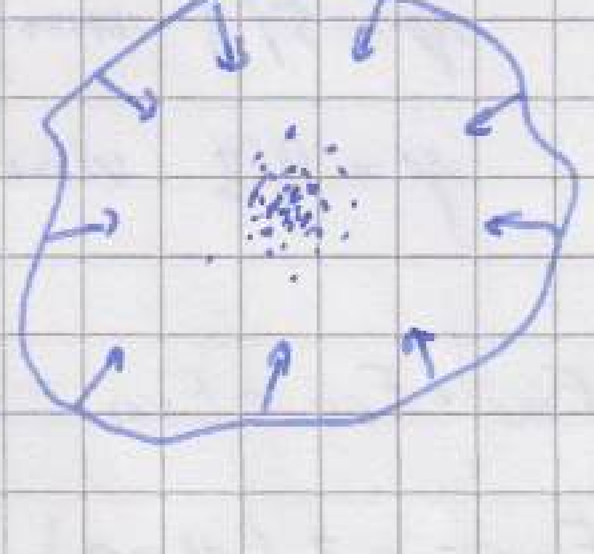
↳ two population:

globular cluster	Pop II	• deficient in heavy elements • compact aggregations • very old
open cluster	Pop I	• loose collections of stars • rich on heavy elements (metals) • comparatively young stars

Conditions for gravitational collapse

Initial point: sufficiently massive and compact cool gas cloud

↳ gravity forces overwhelming internal pressure



The cloud becomes bound if the magnitude of the gravitational potential energy is larger than the internal kinetic energy.

gravitational potential energy: $E_{GR} = -f \frac{GM^2}{R}$

$f \approx \frac{3}{5}$ for a spherical cloud of uniform density

$M = N \cdot \overline{m}$

N: number of particles; $\overline{m}$ ↳ average mass

thermal kinetic energy: $E_{KE} = \frac{3}{2} NkT$

(each particle contributes $\frac{3}{2}kT$)

Condition for the onset of condensation

$$|E_{GR}| > E_{KE}$$

Critical mass: for $f = 1$ and $M = N\bar{m}$

$$\boxed{M_J = \frac{3}{2}\frac{kT}{G\bar{m}} R}$$ Jeans mass

Critical density: $R = \frac{2G\bar{m}M}{3kT}$ $V = \frac{4}{3}\pi R^3$ $\rho = M/V$

$$\boxed{\rho_J = \frac{3}{4\pi M^2}\left[\frac{3kT}{2G\bar{m}}\right]^3}$$ Jeans density

Example:
- cloud of molecular hydrogen at a temperature of 20K

  mass: 2×10^{33} kg ($\sim 1000\, M_\odot$)

  critical density: $\sim 10^{-22}$ kg m^{-3}

- cloud of $1 M_\odot$

  critical density: $\sim 3.8 \cdot 10^{-22}$ kg m^{-3}

A large cloud will be able to fragment into many parts with masses comparable the sun mass

↳ fragmentary gravitational collapse

- each fragment ⇒ protostar
- all protostars ⇒ open star cluster

The contraction of a protostar

Protostar: $R \sim 10^{15}$ m ($\sim$ 1 million × larger as sun)

↳ collaps in free fall

time ↓
- grav. Energy not converted into therm. Energy
- dissociation of hydrogen molecules
- late ionization of hydrogen atoms

Dissociation of hydrogen molecules and ionization of hydrogen atoms

dissociation energy H_2: $\epsilon_D = 4.5\ eV$

ionization energy H: $\epsilon_I = 13.6\ eV$

m_H mass of a hydrogen atom

Energy reservoir of a Protostar with mass M:
(without nuclear energy)

$$\underbrace{\frac{M}{2m_H}}_{N_{H_2}} \epsilon_D + \underbrace{\frac{M}{m_H}}_{N_H} \epsilon_I$$

$\Rightarrow$ this quantity of energy is supplied by the gravitational collapse from an initial Radius R_1 to a final Radius R_2:

$$\frac{GM^2}{R_2} - \frac{GM^2}{R_1} \approx \frac{M}{2m_H}\epsilon_D + \frac{M}{m_H}\epsilon_I$$

Example: $M = 1 M_\odot \Rightarrow 3 \times 10^{39}\ J$ $(R_1 \approx 10^{15} m;\ R_2 \approx 10^{11} m)$

(vom 10.000-fachen auf das 100-fache des Sonnenradius)

$\rightarrow$ free fall time: $\approx 21000\ a$

When most of the hydrogen is ionized the protostar becomes increasingly opaque

↳ heating of the gas $\rightarrow$ internal pressure rises

↳ rapid collapse $\rightarrow$ slow contraction

The thermal kinetic energy of the hydrogen ions and electrons in the Protostar at an internal temperature T is

$$E_{KE} \approx \frac{M}{m_H} 3kT$$

and the gravitional energy at the end of the periode of rapid collapse is (because $R_1 >> R_2$)

$$E_{GR} \approx -\frac{GM^2}{R_2} \approx -\left[\frac{M}{2m_H}\epsilon_D + \frac{M}{m_H}\epsilon_I\right]$$

and for a sphere of ideal gas (Virial-Theorem)

$$2E_{KE} + E_{GR} = 0$$

A protostar approaches hydrostatic equilibrium at a temperature given by

$$kT \approx \frac{1}{6}\left[2\epsilon_D + \epsilon_I\right] \approx 3 \cdot 10^4 K$$

$$kT \approx \frac{1}{12}\left[\epsilon_D + 2\epsilon_I\right] \approx \underbrace{2.6\ eV}_{k \cdot T \Rightarrow 30000 K}$$

This estimate is independent of the mass of the protostar!

→ the subsequent slow contraction is governed by the opacity of the ionized interior

↳ opacity controls the rate at which energy is lost as radiation from the stars surface

→ contraction duration: 10^7 to 10^8 years

→ we can use the virial theorem because the protostar remains close to a state of hydrostatic equilibrium !

↳ $1/2\ E_g$ → radiated from the surface

$1/2\ E_g$ → stored as kinetic energy in stars interior

→ interior temperature increase → $\underbrace{\text{thermonuclear fusion of H}}$

protostar ceases to contract

⇓

Stability

⇓

Stardom is reached ⇒ main sequence state

Conditions for stardom

Not all self-gravitating bodies achieve stardom

↳ Brown dwarfs, Giant planets ...

Two kinds to opposite gravitation: • gas pressure

• pressure of a degenerate gas (electrons, neutrons)

Example: degenerate electron gas

A degenerate electron gas resists compression, not because of random thermal energy of the electrons (like the ions in a classical gas), but because the total kinetic energy of the electrons has a minimum value which increases as the density rises.

- de Broglie wavelength of electrons: $\lambda = \frac{h}{p}$
- kinetic energy of an e^- in classical gas: $E_{ki} \sim kT$

$$p \sim \sqrt{m_e kT}$$

typical wavelength: $\lambda \approx \frac{h}{\sqrt{m_e kT}}$

classical condition: the average separation between the electrons has to be large compared with λ

$$\varrho \ll \frac{\underbrace{\bar{m}}_{\text{average mass of the particles in the ionized gas}}}{\lambda^3} \approx \bar{m}\,\frac{(m_e kT)^{3/2}}{h^3}$$

ionized hydrogen: $\underbrace{\bar{m} = 0.5\,\text{amu}}_{\text{average mass of } p+e^-}$

$$E_{GR} = -f\,\frac{GM^2}{R} \qquad E_{KE} = \frac{3}{2}NkT$$

hydrostatic equilibrium: $2E_{KE} + E_{GR} = 0$

$$\Downarrow$$

$$kT \approx \frac{GM\bar{m}}{3R} \approx G\bar{m}M^{2/3}\varrho^{1/3}$$

because: $N = M/\bar{m}$ $\quad \sqrt[3]{\varrho} = \frac{1}{R}\sqrt[3]{\frac{3M}{4\pi}} \Rightarrow \sqrt[3]{\frac{4}{81}\pi}\cdot G\bar{m}M^{2/3}\varrho^{1/3}$

Recognizing: the temperature is proportional to $\varrho^{1/3}$

↳ this will be the case as long as the density is low enough to satisfy

$$\varrho \ll \frac{(m_e kT)^{3/2}}{h^3}$$

In this case the electrons behave a classical gas

But when the density reaches the value

$$\varrho \approx \bar{m} \frac{(m_e kT)^{3/2}}{h^3}$$

Quantum mechanics becomes important → electrons degenerate

Example: $T \sim 10^6\,K \quad \varrho \approx 250\ kg\,m^{-3}$

$T \sim 10^7\,K \quad \varrho \approx 8000\ kg\,m^{-3}$

In the degenerate gas, the temperature of the gas is no longer increases markedly if its compressed

↳ degenerating pressure is independently from temperature

Estimate the temperature at which the free electron gas in the interior of the contracting protostar become degenerate:

$$\varrho \approx \bar{m} \frac{(m_e kT)^{3/2}}{h^3} \Rightarrow kT \approx G\bar{m} M^{2/3} \varrho^{1/3}$$

$$kT \approx G\bar{m}M^{2/3}\ \bar{m}^{1/3}\ \frac{(m_e kT)^{1/2}}{h}$$

⇓ (square)

$$\boxed{kT \approx \left[\frac{G^2\bar{m}^{8/3} m_e}{h^2}\right] M^{4/3}}$$

M plays the key role!

Around this temperature, degenerate electrons begin to resist compression and further contraction under gravity no longer causes the temperature to rise.

<u>Example:</u> $M \approx 1 M_\odot \rightarrow kT \approx 1\,keV \hat{=} 6.6 \cdot 10^7\,K$

Detailed calculations indicate that the minimum mass needed for thermonuclear ignition is about $0.08\,M_\odot$

(lower) Limit for $\underbrace{\text{brown dwarfs}}$

Stabilized by the pressure of a degenerate electron gas

<u>Upper limit of M:</u>

→ gas particles become ultra-relativistic
→ radiation pressure become significantly
→ <u>limits:</u> $50 \ldots 100\,M_\odot$

1.4. The Sun

→ describe: Standard Solar Model (Bahcall et al.)

<u>Important parameters</u>

$M_\odot = 1.99 \times 10^{30}\,kg$ $\quad R_\odot = 6.96 \times 10^8\,m$

$L_\odot = 3.86 \times 10^{26}\,W$ $\quad T_{\odot_e} = 5780\,K$

<u>Central parameters</u>

$\varrho_c = 1.48 \times 10^5\,kg\,m^{-3}$ $\quad T_c = 15.6 \times 10^6\,K$

$P_c = 2.29 \times 10^{16}\,Pa$

<u>Age</u> $\quad t_\odot \approx 4.55 \times 10^9\,a$

Pressure, density and temperature

Free Fall time sun: $t_{FF} = \left[\frac{3\pi}{32 G \underbrace{\langle \varrho \rangle}_{1.4\times 10^{3}\,kg\,m^{-3}}}\right]^{1/2} \approx 30\,min$

The sun has been close to hydrostatic equilibrium for at least 4.5 billion years

↳ average pressure: (from virial theorem) $\langle P \rangle = -\frac{1}{3}\frac{E_{GR}}{V} \approx \frac{G M_\odot}{4\pi R_\odot^4} \approx 10^{14}\,Pa$ a)

↳ average pressure: (from kinetic theory) $\langle P \rangle = \frac{\langle \varrho \rangle}{\underbrace{\bar{m}}_{0.5\,amu}} k \underbrace{T_I}_{\text{typical internal Temperature}}$ b)

$\underbrace{1\,amu}_{\text{atomar mass unit}} = 1.6605 \times 10^{-27}\,kg \Rightarrow \frac{m_a(^{12}C)}{12}$ $m_a = A \cdot \underbrace{m_u}_{6.022\times 10^{23}\,mol^{-1}\text{ Avogadro constant}}$

Standard solar model: 71% H; 27% He; 2% metals

↳ when fully ionized: $\bar{m} \approx 0.61\,amu$

Estimate the typical temperature of the sun

From (a) and (b): $kT_I \approx \frac{G M_\odot \bar{m}}{3 R_\odot} \approx 0.5\,keV$

$T_I \approx 7.7 \times 10^{6}\,K$

Solar radiation

Luminosity L = total power radiated by the sun

$L_\odot \approx 4 \cdot 10^{26}\,W$

Sun as black body: $L_\odot = 4\pi R_\odot^2 \sigma \underbrace{T_E^4}_{\text{effective temperature}}$ ↳ ≈ surface temp.

$\sigma = 5.67 \cdot 10^{-8}\,W m^{-2} K^{-4}$ (Stefan's const.)

Since $kT_E \approx 0.5\,eV$ the bulk of radiation is in the visible part of the electromagnetic spectrum

$$\lambda_m \cdot \underbrace{T_E}_{\approx 6000K} = 2.9 \times 10^{-3}\,mK \qquad \lambda_m = 480\,nm$$

$\Rightarrow$ T_E is three orders of magnitude less than the typical interior temperature of T_I ($\approx 6\,000\,000\,K$)

↳ centre is hotter $\Rightarrow$ radiation transfer to surface

Without opacity → sun would appear to be a black body radiator at temperature T_I

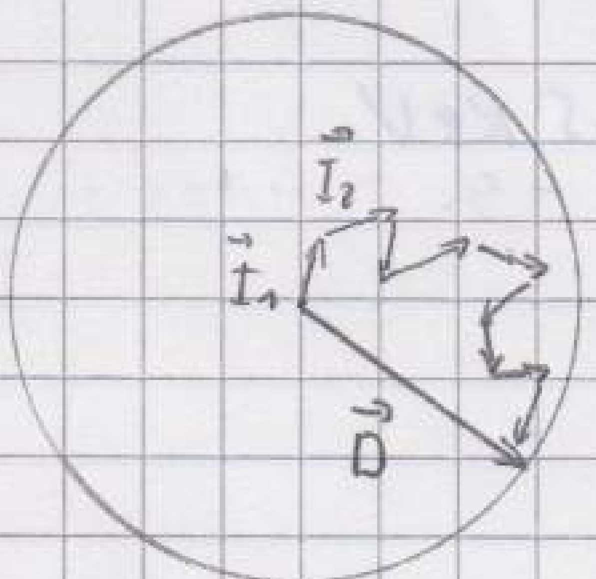
↳ λ_m is the X-ray region

($T_I \approx 6 \cdot 10^6 K \rightarrow \lambda_m \approx 50\,nm$)

The radiation inside the sun is continually scattered, absorbed and emitted by electrons and ions. A temperature gradient is set up and the radiant energy slowly diffuses towards the surface, where it escape as visible radiation.

Random walk through the sun interior

l is the free path length of a photon within the sun



$$\vec{D} = \vec{I}_1 + \vec{I}_2 + \ldots + \vec{I}_n$$

$\vec{I}_k$ photon random pathways

entire pathway:

$$D^2 = l_1^2 + l_2^2 + \ldots + l_N^2 + \underbrace{2(\vec{I}_1\vec{I}_2 + \vec{I}_1\vec{I}_3 + \ldots)}_{\text{sum of all cosine of direction } = 0}$$

$D^2 = N l^2$ time per random step: l/c

$\Rightarrow$ random walk escape time: $t_{RW} \approx \dfrac{R_\odot^2}{c\,l}$

(radiative diffusion)

by $l \approx 0.7$ mm: typical internal temperature T_I is related to the effective surface temperature T_{EF}

the sun is very opaque!

$$T_E \approx \left[\frac{l}{R_\odot} \right]^{1/4} T_I$$

The typical time for radiation to diffuse from the centre and escape from the sun is about 73000 years.

How would the luminosity of a sun-like star depends on its mass?

$$L_\odot \approx 4\pi R_\odot^2 \sigma T_I^4 \frac{l}{R_\odot} \approx \frac{(4\pi)^2 \sigma}{3^5 k^4} G^4 \bar{m}^4 \langle \varrho \rangle l M_\odot^3$$

$$L_\odot \sim M_\odot^3$$

The luminosity of a sun-like star is a rapidly increasing function of its mass.

⇒ radiative diffusion restricts the flow of radiation and prevents the sun from losing heat catastrophically. It determines the luminosity and hence the rate at which energy must be released by thermonuclear fusion at the centre of the sun.

Thermonuclear fusion in the sun

→ dominant reaction: proton-proton-chain

$p + p \to d + e^+ + \nu_e$ (very slow, weak interaction)

$p + d \to {}^3He + \gamma$

${}^3He + {}^3He \to {}^4He + p + p \quad + 26$ MeV

→ the first reaction in the chain governs the rate at which energy is released by the entire chain

⇒ on average, each kilogram of the solar matter generates only 0.2 mW.

↳ power, generated from human body is 10000 times large!

⇒ consumption rate of protons: $\underbrace{4 \times 10^{38}\ p/sec}$

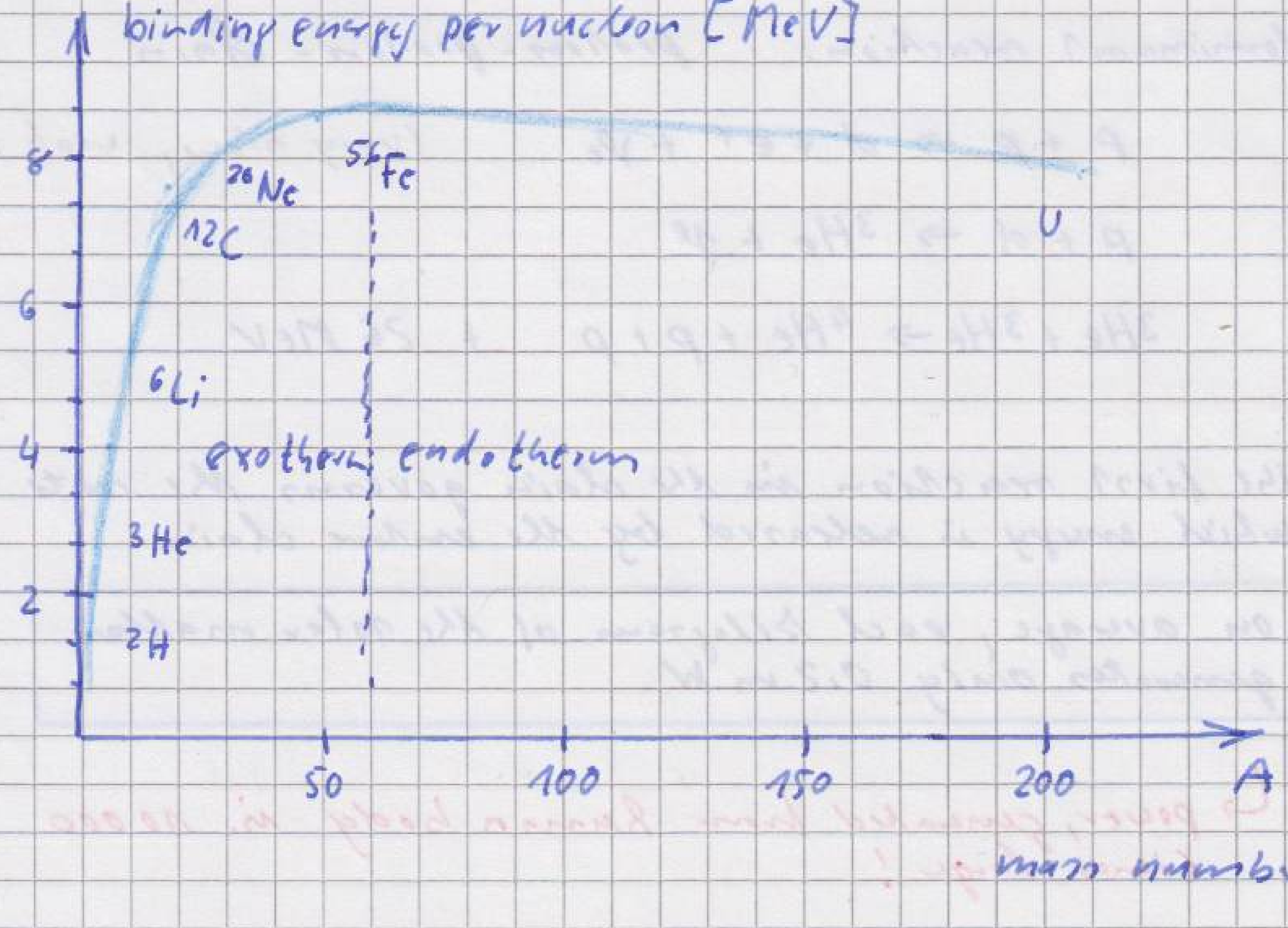
leads to emission of 2×10^{38} neutrinos per second

1.5. Stellar nucleosynthesis

Stellar evolution involves the release of gravitational potential energy through contraction, with pauses whenever nuclear fuels are ignited so as to supply the energy flow from the surface of the star.

contraction → stable nuclear fusion → instability - contraction → stable...
→ Star evolution → sequence of thermonuclear stages

Process	Fuel		Products	
Hydrogen burning	H	→	He	1×10^7 K
Helium burning	He	→	C, O	1×10^8 K
Carbon burning	C	→	O, Ne, Na, Mg	5×10^8 K
Neon burning	Ne	→	O, Mg	1×10^9 K
Oxygene burning	O	→	Mg to S	2×10^9 K
Silicon burning	Si	→	Fe and nearby elements	3×10^9 K



Stellar mass and the extent of thermonuclear fusion

minimal mass: internal temperature increase → ceased if the electrons degenerate
↳ Brown dwarf $0.08\, M_\odot$

⇒ $\boxed{\text{The mass of a contracting star determines the maximum temperature achievable and hence which thermonuclear fusion stages can be reached}}$

$T_{max} \sim M^{4/3}$

it gives two mechanism for hydrogen burning
- p-p-chain → important in stars like the sun
- Bethe-Weizsäcker → for stars with masses $> M_\odot$ important

$\underbrace{\qquad\qquad}$

when this ceased (nach ca. 10% Wasserstoffverbrand)

⇓

core contraction (temperature increased) $T_c \approx 10^8\,K$, $\varrho_c \approx (10^5 .. 10^8\, kg\, m^{-3})$

⇓

- helium-burning $\underbrace{4He + {}^4He + {}^4He \rightarrow {}^{12}C}_{\text{triple } \alpha\text{-process}}$

(+ H-shell burning)

⇓

outer layer of the star expanded ⇒ red giant state

⇒ To achive these condidtions the initial mass of the star must exceed a value of about $0.5\, M_\odot$

- oxygene burning → ${}^4He + {}^{12}C \rightarrow {}^{16}O + \gamma$

(neon burning) → ${}^4He + {}^{16}O \rightarrow {}^{20}Ne + \gamma$

⇒ interior: wandering burning shells
↳ building of a carbon-oxygene core

- Stars with a mass greater than $8 M_\odot$ or thereabouts, can progress beyond helium burning and ignite carbon at a temperature of $\sim 5 \times 10^8 K$ to form such elements as Ne, Na and Mg.

- If the temperature exceeds $10^9 K$, carbon burning can be followed by the photodisintegration of Ne to produce O and He-nuclei;

  the He-nuclei are then captured by undissociated Ne-nuclei to form Mg

- Oxygene burning can then take place at about $2 \times 10^9 K$ to produce elements between Mg and S.

- Stars with a mass greater than $11 M_\odot$, or thereabouts, are able to achieve the high temperature of about $3 \times 10^9 K$ which needed to ignite silicon burning, the final stage of thermonuclear fusion.

  ↳ leads to the formation of nuclei near Fe

The mass of a star governs the extent to which it converts hydrogen to heavier elements.

0.1 ... 0.5 $M_\odot$	hydrogene burning
0.5 ... 8 $M_\odot$	hydrogene + helium burning
8 ... 11 $M_\odot$	... to carbon burning
greater 11 $M_\odot$	... ignite every stage of thermonuclear fusion

Neutron capture

⇒ We search a mechanism to create elements heavier than iron.

↳ These elements owe their existence to neutron capture

- n electrically neutral → easily captured by nucleus
- presence of neutrons can lead to the production of neutron-rich isotopes

  β-decay → form stable elements

s-process : production of neutrons in evolved stars is normally a slow process

r-process : neutron production may become very rapid during the final stage of evolution of a very massive star

↳ neutron capture very effective before onset of β-decay

↳ this process is rapid

The types of nuclei produced by these two processes differ significantly.

⇒ no element beyond bismuth with Z = 83 can be formed by the s-process

1.6 The stellar life cycles

The stellar life cycle is a history of the cosmic matter cycle.

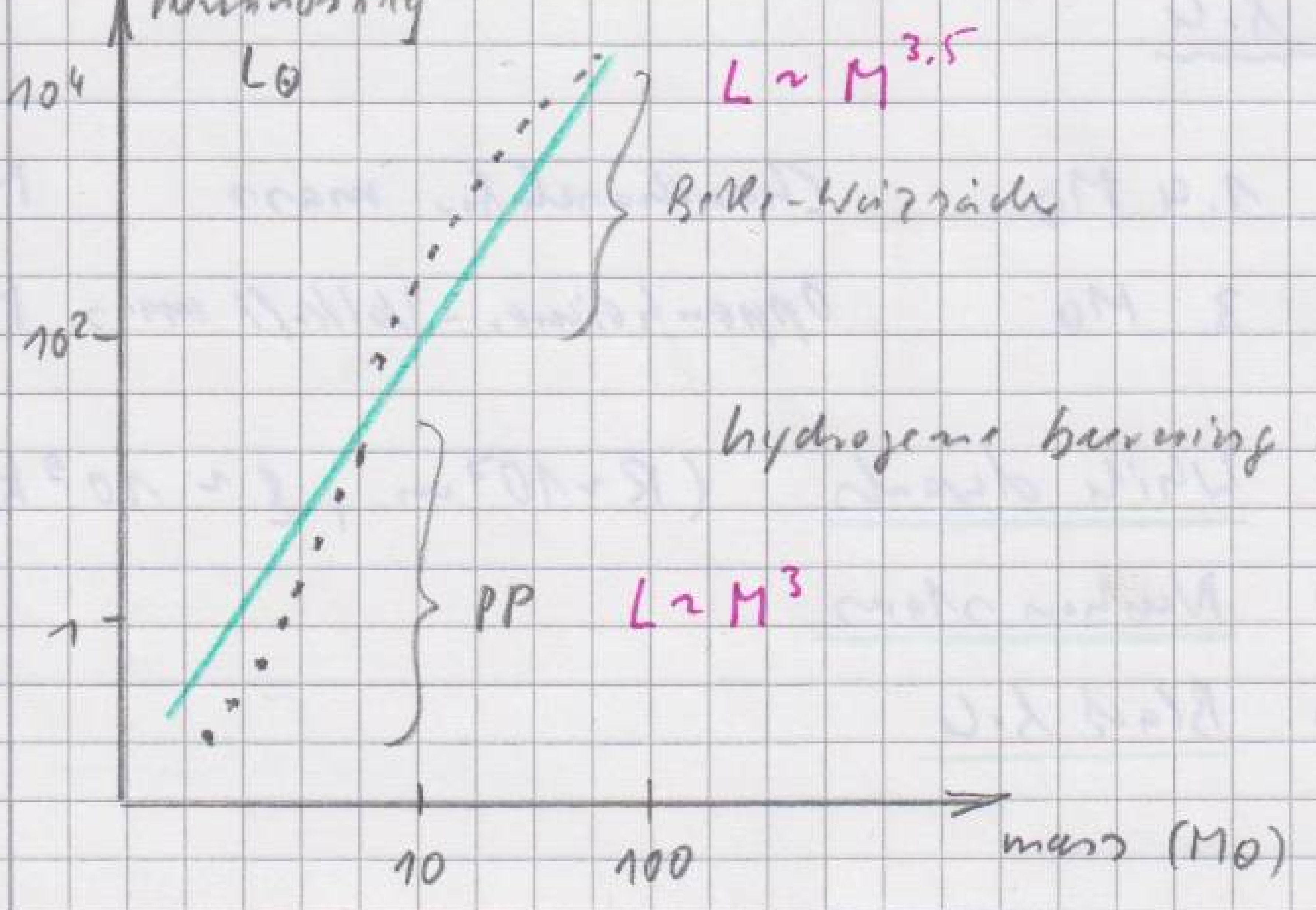
Big Bang (H, He) → first star generation → (H, He, C, O ...) → next star generation ...

interstellar matter → stars → interstellar matter

↳ enriched with „metals"

Rate of stellar evolution

mass-luminosity relation of main sequence stars



The rapid increase of luminosity with mass has an important implication:

$\boxed{\text{massive stars have short lives despite their greater resource of fuel}}$

hydrogen burning lifetime: $\sim M^{-2}$ high mass stars

$\sim M^{-2.5}$ low mass stars

Example: $10\,M_\odot$ → main sequence ~ 100 million years

$0.5\,M_\odot$ → " " ~ 50 billion years

⇒ $\boxed{\text{more massive stars evolve more quickly}}$

The endpoints of stellar evolution

The ultimate fate of a star depends crucially on the mass that remains in the central core when nuclear fusion can no longer maintain the pressure needed to prevent gravitationally contraction.

↳ non thermal source of pressure

↳ degenerate e^--gas: White dwarfs

↳ degenerate neutron gas Neutron stars

↳ no source of pressure

↳ Black hole

critical mass: $1.4\,M_\odot$ Chandrasekhar mass M_{CH}

$3\,M_\odot$ Oppenheimer - Volkoff mass M_{OV}

$M < M_{CH}$ ⇒ White dwarfs ($R \sim 10^7\,m$, $\varrho \sim 10^9\,kg\,m^{-3}$)

$M_{CH} < M < M_{OV}$ ⇒ Neutron stars

$M > M_{OV}$ ⇒ Black hole

Abundances of the chemical elements

The chemical elements observed in the solar system are largely a reflection of the combined effect of nucleosynthesis during the big bang and of nucleosynthesis during stellar evolution of earlier generations of nearby stars.

↳ relative abundances of elements

- dominance of hydrogen and helium (left-over from big bang)
- a distinct lack of abundance between helium and carbon → nuclear physics effect (absence of stable mass 5 and mass 8 atomic nuclei)
- peaks corresponding to the major products of stellar nucleosynthesis → C, O, Ne, Si and elements near Fe.

1.7. The Hertzsprung-Russell diagram

→ the most important diagram for the star physics.

Luminosity

What we see: apparent magnitude of stars

$$\frac{I_1}{I_2} = 10^{0.4\Delta m} \qquad m_1 - m_2 = -2.5 \log_{10} \underbrace{(f_1/f_2)}_{\text{energy flux}}$$

$\Rightarrow \Delta m = 2.5$ for $f_2 = 10 f_1$

→ energy flux is inversely proportional to the square of the distance of the star:

$$m_1 - m_2 = -2.5 \log_{10} \underbrace{(d_2^2/d_1^2)}_{\text{distances}} = 5 \log_{10} (d_1/d_2)$$

Visual magnitude m_V (only visual spectral range)

bolometric " m_B (entire spectral range)

The difference $M_B - m_V$ is called "bolometric correction"

The absolute bolometric magnitude corresponds to the brightness of a star as measured at a distance of 10 parsec (entire spectrum)

$1\,pc = 3.086 \times 10^{16}\,m$ ($\hat{=}$ 3.26 light years)

If we known the distance of a star, we can calculate the luminosity from the apparent magnitude

↳ absolute bolometric magnitude M_B

$$\boxed{M_B = -2.5 \log_{10} (L/L_\odot) + 4.72}$$

$L_\odot = 4 \times 10^{26}\,W$ $\qquad$ (M_B(sun) $= +4.72$)

luminosity range $10^{-4} L_\odot \ldots 10^{6} L_\odot$

↳ bolometric magnitude varies $+15 \ldots -10$

Surface Temperature

The effective surface temperature of a star, T_E, is defined as the temperature of the black body of the same size which would give the same luminosity.

$$\boxed{L = 4\pi R^2 \sigma T_E^4}$$

sun: $T_E \approx 6000\,K$

"color" of a star is a alternative measure of its surface temperature

↳ measuring the apparent magnitude using filters

V: transmit by 550 nm $\qquad$ B: by 440 nm

Colour index B-V

negative → blue star

positive → red star

The colour temperature corresponds to the temperature of the Planck black body spectrum which gives the observed value of the colour index.

Another method ⇒ spectral lines in absorption spectra

↳ spectral sequence is a temperature sequence

O-B-A-F-G-K-M

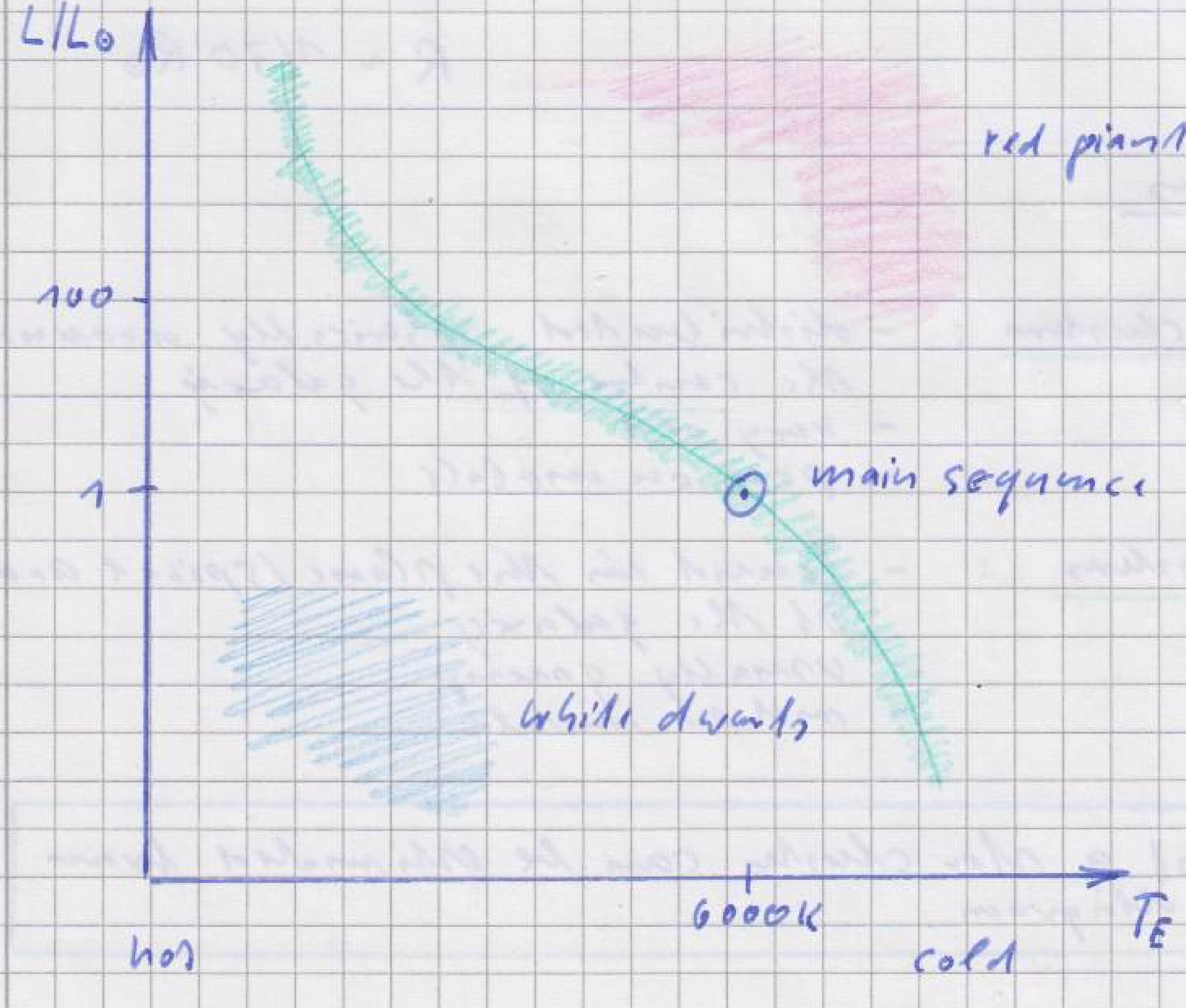
hot ⟶ cold

30.000 K 3.000 K

Luminosity and surface temperature

→ luminosity and the effective temperature are correlated

↳ illustrated ⇒ Hertzsprung-Russell-Diagram



An Hertzsprung-Russell diagram provides a snapshot of stars at different stages of their evolution.

- main sequence : - hydrogen burning stars
  - 80-90% of observed stars
  - hottest stars: blue supergiants
  - coolest stars: red dwarfs
  - mass range: $50 M_\odot \dots 0.1 M_\odot$

=> a star does not evolve along the main sequence !

- red giant region : - hydrogen is depleted in the central region
  - core contraction → temperature increases
  - start helium-burning (3α process)
  - outer layer expands enormically
  - surface temperature decreases
  - luminosity increases

  typical values: $L \sim 1000\, L_\odot$
  $T_E \sim 4000\, K$
  $R \sim 70\, R_\odot$

- white dwarfs : - end stages of intermediate mass stars
  - only core, degenerated
  - will only cool and fade away

  typical values: $L \sim 1/100\, L_\odot$
  $T_E \sim 16000\, K$
  $R \sim 1/70\, R_\odot$

Star clusters

- globular clusters : - distributed spherically around the centre of the galaxy
  - very old
  - poor on metals

  } Pop. II

- open clusters : - found in the plane (spiral arms) of the galaxy
  - usually young
  - rich on metals

  } Pop. I

$\boxed{\text{The age of a star cluster can be estimated from its H-R-diagram.}}$

↳ „length" of the main sequence

Exercises

Consider a sphere of mass M and radius R. Calculate the gravitational potential energy of the sphere assuming (a) a density which is independent of the distance from the centre, and (b) a density which increases towards the centre according to

$$\varrho(r) = \varrho_c\,(1 - r/R).$$

In both cases, (a) and (b), write down the average internal pressure needed for hydrostatic equilibrium, and determine how the pressure within the sphere depends on the distance from the centre.

density function $\quad \varrho(r) = \varrho_c \left(1 - \left(\frac{r}{R}\right)^2\right) \quad$ case A

- radial mass function

$$m(r) = 4\pi \int_0^r \varrho_c \left(1 - \frac{r^2}{R^2}\right) r^2\,dr = \frac{4\pi\varrho_c\,(5R^2 - 3r^2)\,r^3}{15R^2}$$

- average density $\quad \bar{\varrho} = \frac{3M}{4\pi R^3} \qquad m(R) = M = \frac{8\varrho_c \pi R^3}{15}$

$$\varrho_c = \frac{15M}{8\pi R^3} = \frac{5 \cdot 3M}{2 \times 4\pi R^3} = \frac{5}{2}\bar{\varrho}$$

- potential energy $\quad E_p = -\int_0^M \frac{G\,m(r)}{r}\,dm$

$$dm = 4\pi\varrho(r)\,r^2\,dr = 4\pi\varrho_c\left(1 - r^2/R^2\right) r^2\,dr$$

$$E_p = -\int_0^R \frac{G\,4\pi\varrho_c\,(5R^2 - 3r^2)\,r^3}{15R^2 r} \cdot 4\pi\varrho_c\left(1 - \frac{r^2}{R^2}\right) r^2\,dr$$

$$= -\frac{16\pi^2 G\varrho_c^2}{15}\int_0^R \frac{(5R^2 - 3r^2)\,r^4}{R^2}\left(1 - \frac{r^2}{R^2}\right) dr$$

$$= -\frac{16\pi^2 G\varrho_c^2 \cdot 4R^5}{315} = -\frac{16\pi^2 G\,25\bar{\varrho}^2 \cdot 4R^5}{4 \times 315}$$

$$= -\frac{80 G \pi^2 \bar{\varrho}^2 R^5}{63} = -\frac{80 \pi^2 G\, 9 M^2 R^5}{63 \cdot 16 \pi^2 R^6} = -\frac{5}{7}\frac{G M^2}{R}$$

density function: $\varrho(r) = \varrho = \text{const.}$ case B

$$m(r) = 4\pi\varrho \int_0^r r^2\,dr = \frac{4}{3}\pi\varrho r^3$$

potential energy

$$E_p = -\int_0^R \frac{G 4\pi\varrho r^3 \cdot 4\pi\varrho r^2}{3r}\,dr = -\frac{16}{3}\pi^2\varrho^2 G \int_0^R r^4\,dr$$

$$= -\frac{16\pi^2 G \varrho^2 R^5}{15} = -\frac{16\pi^2 G \cdot 9 M^2 R^5}{15 \cdot 16 \pi^2 R^6} = -\frac{3}{5}\frac{G M^2}{R}$$

density function: $\varrho(r) = \varrho_c\left(1 - \frac{r}{R}\right)$ case C

$$m(r) = 4\pi\varrho_c \int_0^r \left(1 - \frac{r}{R}\right) r^2\,dr = \frac{4\pi\varrho_c r^3 (4R - 3r)}{12R}$$

$$= \frac{\pi\varrho_c r^3 (4R - 3r)}{3R}$$

$$m(R) = M = \frac{\pi\varrho_c R^3}{3} \qquad \varrho_c = \frac{3M}{\pi R^3} \qquad \bar{\varrho} = \frac{3M}{4\pi R^3}$$

$$\Rightarrow \varrho_c = \frac{1}{4}\bar{\varrho}$$

potential energy

$$E_p = -\int_0^M \frac{G\,m(r)}{r}\,dm \qquad dm = 4\pi\,\varrho_c\left(1-\frac{r}{R}\right)r^2\,dr$$

$$= -\int_0^R \frac{G\pi\varrho_c r^3(4R-3r)}{3Rr}\,4\pi\varrho_c\left(1-\frac{r}{R}\right)r^2\,dr$$

$$= -\int_0^R \frac{G\cdot 4\pi^2\varrho_c^2 r^4(4R-3r)\left(1-\frac{r}{R}\right)}{3R}\,dr$$

$$= -\frac{4\pi^2 G\varrho_c^2}{3}\int_0^R \frac{r^4(4R-3r)\left(1-\frac{r}{R}\right)}{R}\,dr$$

$$= -\frac{4\pi^2 G\varrho_c^2\,13R^6}{3\cdot 210\,R} = -\frac{4\pi^2 G\cdot 9M\cdot 13R^6}{3\cdot 210\cdot\pi^2 R^7}$$

$$= -\frac{26}{35}\frac{GM^2}{R}$$

average pressure (hydrostatic equilibrium)

A) $$\langle P\rangle = -\frac{E_p}{3V} = -\frac{E_p}{4\pi R^3}$$

$$\langle P\rangle = \frac{5}{28}\frac{GM^2}{\pi R^4} \qquad \text{für } \varrho(r) = \varrho_c\left(1-\frac{r^2}{R^2}\right)$$

B) $$\langle P\rangle = \frac{5}{20}\frac{GM^2}{\pi R^4} \qquad \text{für } \varrho(r) = \varrho = \text{const.}$$

C) $$\langle P\rangle = \frac{13}{70}\frac{GM^2}{\pi R^4} \qquad \text{für } \varrho(r) = \varrho_c\left(1-\frac{r}{R}\right)$$

pressure function inside the star

B) $\varrho(r) = \varrho = \text{const.}$ $\qquad m(r) = \frac{4}{3}\pi\varrho r^3$

$$\frac{dP}{dr} = -\frac{G\,m(r)\,\varrho(r)}{r^2} = -\frac{4}{3}\pi G \varrho^2 r$$

$$P(r) = -\frac{4}{3}\pi G \varrho^2 \int_0^r r\,dr = \underline{\underline{-\frac{2}{3}\pi G \varrho^2 r^2}}$$

and with $\bar{\varrho} = \frac{3M}{4\pi R^3}$ $\Rightarrow$ $P(r) = -\frac{3}{8}\frac{GM^2}{\pi R^6} r^2$

$$= \underline{\underline{-\frac{3}{8}\frac{GM^2}{R^4}}}$$

A) $\varrho(r) = \varrho_c\left(1 - \frac{r^2}{R^2}\right)$ $\qquad m(r) = \frac{4\pi\varrho_c(5R^2 - 3r^2)r^3}{15R^2}$

$$\frac{dP}{dr} = -\frac{G\,m(r)\,\varrho(r)}{r^2} = -\frac{4\pi G\varrho_c(5R^2-3r^2)r^3}{15R^2 r^2}\varrho(r)$$

$$P(r) = -\frac{4\pi G\varrho_c\varrho(r)}{15}\underbrace{\int_0^r \frac{(5R^2-3r^2)r}{R^2}\,dr}_{\frac{(10R^2-3r^2)r^2}{4R^2}}$$

$$\varrho_c\cdot\varrho(r) = \varrho_c^2\left(1-\frac{r^2}{R^2}\right) \Rightarrow -\frac{(r^2-R^2)(10R^2-3r^2)r^2}{4R^4}$$

$$P(r) = \frac{4\pi G\varrho_c^2}{4\times 15}\,\frac{(r^2-R^2)(10R^2-3r^2)r^2}{R^4}$$

$$= \underline{\underline{\frac{\pi G\varrho_c^2}{15R^4}(r^2-R^2)(10R^2-3r^2)r^2}}$$

$P(R) = 0$

c) $\varrho(r) = \varrho_c \left(1 - \frac{r}{R}\right)$

$$\frac{dP}{dr} = -\frac{G\, 4\pi \varrho_c r^3 (4R - 3r)\, \varrho(r)}{12 R r^2}$$

$$= -\frac{4\pi G \varrho_c \varrho(r)}{12} \int_0^r \frac{r(4R - 3r)}{R}\, dr$$

$$= -\frac{\pi G \varrho_c \varrho(r)}{3R} (2R - r) r^2 = -\frac{\pi G \varrho_c^2 (1 - r/R)}{3R} (2R - r) r^2$$

$$= -\frac{\pi G \varrho_c^2}{3R^3} (r - 2R)(r - R) r^2 \qquad P(R) = 0$$

The globular cluster M13 in Hercules contains about 0.5 million stars with an average mass of about half solar mass. Use the Jeans criteria to check whether this cluster could have been formed in the early universe just after the time when the universe was cool enough for the electrons and nuclei to form neutral atoms; at this time the density of the universe was $\varrho \approx 10^{-27}$ kg m^{-3} and the temperature was $T \approx 10^4$ K

data: $t \approx 10^8$ a after big bang: $\varrho = 10^{-27}$ kg m^{-3}

$T \approx 10^4$ K

M13: 0.5×10^6 stars with $0.5\, M_\odot$

Jeans criteria $\qquad \varrho > \varrho_J = \frac{3}{4\pi M^2} \left[\frac{3kT}{2G\bar{m}}\right]^3$

$\varrho = M/V = \frac{3M}{4\pi R^3} \qquad R = \sqrt[3]{\frac{3M}{4\pi\varrho}} = \sqrt[3]{\frac{3 \cdot 4.97 \cdot 10^{35}\ \text{kg m}^3}{4\pi \cdot 10^{-27}\ \text{kg}}}$

$= 4.92 \cdot 10^{20}$ m (52000 ly)

$M = 0.5 \cdot 10^6 \cdot 0.5\, M_\odot$

$= 4.97 \times 10^{35}$ kg

particle density: (H_2 molecules) $\Rightarrow \rho \approx 10^{-27}\ kg\ m^{-3}$

$\approx 0{,}3\ H_2$ molecules $/m^3$

$\bar{m} = 2m_p$

$$\rho_J = \frac{3}{4\pi\,(4.97\cdot 10^{35})^2}\left[\frac{3\times 1.38\cdot 10^{-23}\cdot 10^4}{2\times 6.67\cdot 10^{-11}\cdot 2\cdot 1.67\cdot 10^{-27}}\right]^3$$

$$= \underline{\underline{7.75\times 10^{-19}\ kg\ m^{-3}}} \quad \Rightarrow \quad \rho < \rho_J$$

To this early times $\rightarrow$ collaps impossible

$$M_J = \frac{3kT}{2G\bar{m}}R = \frac{3\times 1.38\cdot 10^{-23}\cdot 10^4 \times 4.92\cdot 10^{20}}{2G\cdot 2m_p} = 4.56\cdot 10^{38}\ kg$$

$\Rightarrow M < 4.56\times 10^{38}\ kg \rightarrow$ collaps impossible

As the sun evolved towards the main sequence, it contracted under gravity while remaining close to hydrostatic equilibrium, and its internal temperature changed from about 30 000 K to about 6×10^6 K.
Find the total energy radiated during this contraction. Assume that the luminosity during this contraction is comparable to the present luminosity of the sun and estimate the time taken to reach the main sequence.

give: $T_1 = 3\times 10^4\ K$ (end of free fall phase)
$T_2 = 6\times 10^6\ K$ (hydrostatic equilibrium)

search: time duration to achive main sequence (for $1M_\odot$ Star)
radiated energy

Virial theorem: $\quad 2E_K + E_G = 0 \qquad E_K = -\frac{1}{2}E_G$

$E_G \Rightarrow 1/2$ radiated

$E_G \Rightarrow 1/2$ increase temperature

Estimate size of stars

$$R_1 = \frac{2 G m_p M}{3 k T_1} \approx 3.57 \times 10^{11}\, m \quad (514\, R_\odot)$$

$$R_2 = \frac{2 G m_p M}{3 k T_2} \approx 1.75 \times 10^{9}\, m \quad (2.6\, R_\odot)$$

$$E_G = G M^2 \left(\frac{1}{R_2} - \frac{1}{R_1} \right) = \underline{\underline{1.47 \times 10^{41}\, Ws}}$$

$\rightarrow$ radiated: $\quad 7.34 \times 10^{40}\, Ws = E_{ge}$

Luminosity in the contraction phase

with $L_\odot = 3.85 \times 10^{26}\, W$

$$\frac{E_{ge}}{L_\odot} = 1.91 \times 10^{14}\, s \quad (\approx 6 \cdot 10^{6}\, a)$$

A star with a mass of $1 M_\odot$ achives in a time of 6 Million years the main sequence.

The main sequence of the Pleiades cluster of stars consists of stars with mass less than $6 M_\odot$; the more massive stars have already evolved off the main sequence. Estimate the age of the Pleiades cluster.

$M < 6 M_\odot \qquad \Delta t_{ms} \sim M^{-2} \qquad (\text{sun} \sim 10^{10}\, a \; (1 M_\odot))$

$\Delta t_{ms} \sim M^{-2} \qquad \Delta t = t_0 M^{-2} \qquad t_0 = 10^{10}\, a$

$$\Delta t_{ms}(\text{Pleiades}) = 10^{10} \cdot (6 M_\odot)^{-2} \qquad M_\odot = 1$$

$$= \underline{\underline{10^{10}/36 \approx 2.78 \times 10^{8}\, a}}$$

Estimate size of stars

$$R_1 = \frac{2 G m_p M}{3 k T_1} \approx 3.57 \times 10^{11} m \quad (514\, R_\odot)$$

$$R_2 = \frac{2 G m_p M}{3 k T_2} \approx 1.75 \times 10^{9} m \quad (2.6\, R_\odot)$$

$$E_G = G M^2 \left(\frac{1}{R_2} - \frac{1}{R_1} \right) = \underline{\underline{1.47 \times 10^{41}\, Ws}}$$

→ radiated: $7.34 \times 10^{40}\, Ws = E_{ge}$

Luminosity in the contraction phase

with $L_\odot = 3.85 \times 10^{26}\, W$

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$M < 6 M_\odot$ $\quad \Delta t_{ms} \sim M^{-2}$ $\quad$ (sun $\sim 10^{10} a$ $(1 M_\odot)$)

$\Delta t_{ms} \sim M^{-2}$ $\quad \Delta t = t_0 M^{-2}$ $\quad t_0 = 10^{10} a$

$$\Delta t_{ms}(\text{Pleiades}) = 10^{10} \cdot (6 M_\odot)^{-2} \quad M_\odot = 1$$

$$= \underline{\underline{10^{10}/36 \approx 2.78 \times 10^{8} a}}$$

photon energy (visual, $\lambda \sim 540\,nm$)

$$E_\lambda(\lambda = 540\,nm) = \frac{h \cdot c}{\lambda} = \frac{6{,}6261 \times 10^{-34} \cdot 2{,}998 \times 10^{8}}{540 \times 10^{-9}}$$

$$= 3.68 \times 10^{-19}\,Ws$$

$$N_{phot} = \frac{I \cdot A_{eye}}{E_\lambda} = \frac{1{,}03 \times 10^{-10} \cdot 7.85 \cdot 10^{-5}}{3.68 \cdot 10^{-19}}$$

$$= \underline{\underline{2{,}2 \cdot 10^{4} \ (\approx 22000 \text{ photons per second})}}$$

2. Properties of matter and radiation

The conditions in the interior of stars are extreme:

- temperature, pressure, degeneration grade etc.

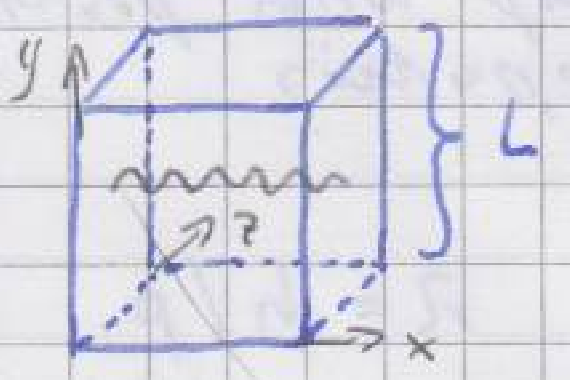
↳ but we can this understand with simple thermodynamics
  ↳ ideal gas

2.1. The ideal gas

> The ideal gas is a large number of particles occupying quantum states whose energy is unaffected by the interaction between the particles.

→ The effects of quantum mechanics and special relativity will often be important

Density of states



cubical box with $V = L^3$

possible quantum states in this box?

- standing waves $\sin \underbrace{k_x}_{\text{component of wave number vector}} x$, $\sin k_y y$, $\sin k_z z$

$$\mathbf{k} = (k_x, k_y, k_z) = \underbrace{(n_x, n_y, n_z)}_{\substack{\text{quantum numbers} \\ \text{(integers)}}} \frac{\pi}{L}$$

The number of quantum states with a wave vector $\mathbf{k}$ with components between k_x and $k_x + dk_x$, k_y and $k_y + dk_y$, k_z and $k_z + dk_z$ is:

$$\left.\left[\frac{L}{\pi}\right]^3 dk_x\, dk_y\, dk_z \right\} \text{"k-space"}$$

<u>k-space</u>: filled with $\underbrace{\text{discrete volume elements}}$

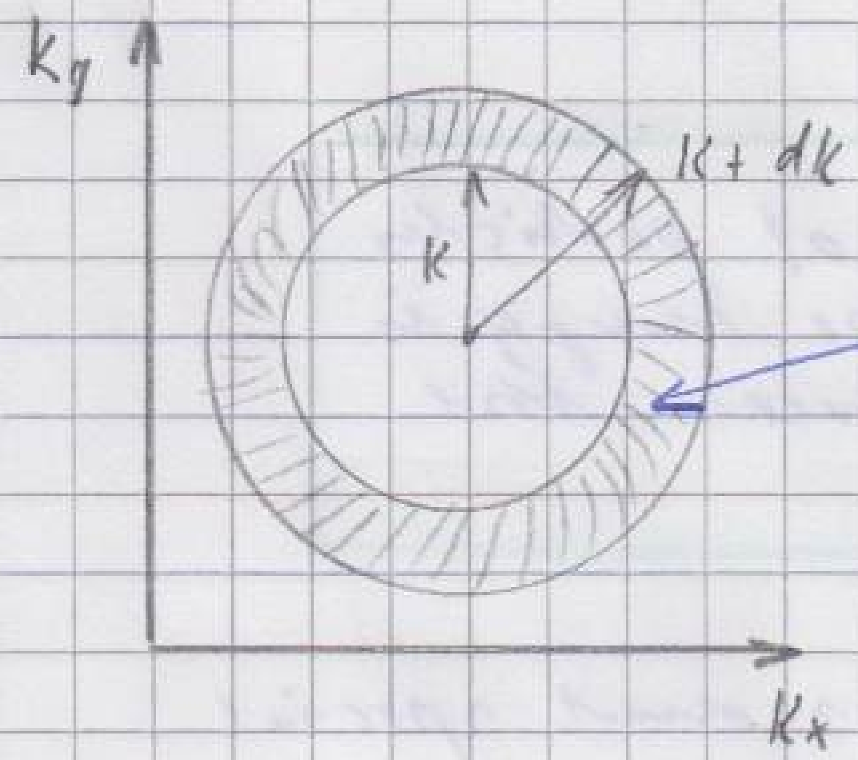
each contains many quantum states with a density of

$$[L/\pi]^3$$

per unit volume

⇒ wave vector **k** with $\underbrace{\text{magnitude between } k \text{ and } k+dk}$

occupies the k-space between two spheres of radius k and k+dk



Volume: $\dfrac{4\pi k^2 \, dk}{8}$ (3 dimensions)

Number of states with wave vector magnitude between k and k+dk:

$$\left[\frac{L}{\pi}\right]^3 \frac{4\pi k^2 \, dk}{8}$$ ⇒ connecting with particle like properties

Quantum particle → de Broglie wave length: $\lambda = h/\underset{\text{momentum}}{p}$

$p = \hbar k \quad p = h/\lambda$ → <u>replace k through p</u>:

Number of quantum states that have a momentum with a magnitude between p and p+dp:

$$\boxed{g(p)\, dp = \frac{V}{h^3} 4\pi p^2 \, dp}$$ (without intrinsic quantum numbers)

and, is g_s the number of independent polarizations of the particles:

$$\boxed{g(p)\, dp = \frac{V}{h^3} 4\pi p^2 g_s \, dp}$$ g_s degree of degeneration

$p, n, e^- \rightarrow$ fermions $\rightarrow g_s = 2$ (have spin $= \pm 1/2$)

$\nu_e \rightarrow$ fermions $\rightarrow g_s = 1$ (only one polarizations)

$\gamma \rightarrow$ bosons $\rightarrow g_s = 2$ (have spin 1, but 2 polarization directions)

Internal energy

The internal kinetic energy of the gas depends on three factors:

- the density of states
- the energy of each quantum state
- the number of particles in each quantum state

$\Rightarrow$ density of states: $g(p)\, dp$

$\Rightarrow$ energy of a particle: $\epsilon_p^2 = p^2c^2 + m^2c^4$

What is the average number of particles in a state with energy ϵ_p?

$$E = \int_0^\infty \epsilon_p \underbrace{f(\epsilon_p)}_{\text{distribution}} \overbrace{g(p)}^{\text{quantum states}} dp$$

Total numbers of particles in the gas:

$$N = \int_0^\infty f(\epsilon_p)\, g(p)\, dp$$

The macroscopic thermodynamic properties of the gas may be described by its temperature T, its pressure P and its chemical potential μ.

$\hookrightarrow$ determine how the internal energy of the gas is changed by a transfer of heat/entropy – by compression or expansion – by particle transfer:

$$dE = \underbrace{T\,dS}_{\text{entropy change}} - P\,dV + \mu\, dN$$

Thermodynamic equilibrium: T, P and μ related by a equation of state

$\Rightarrow$ T, P and μ determine the equilibrium distribution of the particles in the $\underbrace{\text{quantum states}}_{\text{depends on whether the particles are fermions or bosons}}$

a) identical fermions obey Fermi-Dirac-statistics

$$\boxed{f(\epsilon_p) = \frac{1}{\exp[(\epsilon_p - \mu)/kT] + 1}}$$

b) identical bosons obey Bose-Einstein-statistics

$$\boxed{f(\epsilon_p) = \frac{1}{\exp[(\epsilon_p - \mu)/kT] - 1}}$$

Particle ensembles under „normal" conditions: dilute classical gas

$\exp[(mc^2 - \mu)/kT] \gg 1$ (+1; -1 can be neglect) : a gas of bosons and a gas of fermions have similar properties when the occupation of every quantum state is low.

$\Rightarrow$ average number of particles in a quantum state:

$$f(\epsilon_p) \approx \exp[-(\epsilon_p - \mu)/kT] \ll 1$$

by/no quantum effects: $\Rightarrow$ Maxwell-Boltzmann statistics

{ In stars, quantum and classical behavior of electron gases play an essential role }

Pressure in an ideal gas

- derive from statistical mechanics

$$dE = \underbrace{TdS}_{\substack{0 \text{ by constant} \\ \text{Entropy}}} - PdV + \underbrace{\mu\, dN}_{\substack{0 \text{ by constant} \\ \text{particle number}}}$$

$\Rightarrow$ change in internal Energy $\rightarrow$ change in Volume

$$dE = -PdV$$

$$P = -\frac{\partial E}{\partial V} = -\int_0^\infty \frac{d\epsilon_p}{dV} f(\epsilon_p)\, g(p)\, dp$$

$$\frac{d\epsilon_p}{dV} = \frac{d\epsilon_p}{dp} \cdot \underbrace{\frac{dp}{dV}} \qquad V = L^3 \qquad p \sim V^{-1/3}$$

momentum

$$\frac{dp}{dV} = -\frac{p}{3V^{4/3}} \approx -\frac{p}{3V}$$

$$\frac{d\epsilon_p}{dp} = \frac{p \cdot c}{\sqrt{p^2+m^2c^2}} \Rightarrow \frac{d\epsilon_p}{dp} = \frac{pc^2}{\epsilon_p} = \underbrace{v_p}_{\substack{\text{speed of particle with} \\ \text{momentum } p}}$$

$$P = -\int_0^\infty \frac{d\epsilon_p}{dp} \cdot \frac{dp}{dV} f(\epsilon_p)\, g(p)\, dp$$

$\equiv$ classical result $P = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle$

$$= \frac{1}{3V}\int_0^\infty p\, v_p\, f(\epsilon_p)\, g(p)\, dp = \frac{N}{3V}\underbrace{\langle p\, v_p \rangle}_{\substack{\text{average over the } N \\ \text{particles of the gas}}}$$

$$\boxed{P = \frac{N}{3V}\langle p\, v_p \rangle}$$

a) non relativistic particles: $\epsilon_p = mc^2 + \frac{p^2}{2m}$ $\quad v_p = \frac{p}{m}$

$$P = \frac{2N}{3V}\left\langle \frac{p^2}{2m} \right\rangle = \frac{2}{3} \text{ of kinetic energy density}$$

b) ultrarelativistic particles: $\epsilon_p = pc$ $\quad v_p = c$

$$P = \frac{N}{3V}\langle pc \rangle = \frac{1}{3} \text{ of kinetic energy density}$$

The ideal classical gas

> A gas is classical when the average occupation of any quantum state is small $(f(\epsilon_p) \ll 1)$.

$$P = \frac{1}{3V}\exp\left[\frac{\mu}{kT}\right]\int_0^\infty p\, v_p \exp\left[-\frac{\epsilon_p}{kT}\right] g_s \frac{V}{h^3} 4\pi p^2\, dp$$

$$\epsilon_p^2 = p^2c^2 + m^2c^4 \quad \rightarrow \quad d\epsilon_p = v_p\, dp$$

$$\int_0^\infty p^2 \exp\left[-\frac{\epsilon_p}{kT}\right] v_p\, dp = -kT\int_0^\infty p^2 \underbrace{d\left(\exp\left[-\frac{\epsilon_p}{kT}\right]\right)}_{-\frac{1}{kT}d\left(\exp\left[-\epsilon_p/kT\right]\right)}$$

Integration by parts

substitute

$$\int_0^\infty p^2 \exp\left[-\frac{\epsilon_p}{kT}\right] v_p\, dp = 3kT\int_0^\infty \exp\left[-\frac{\epsilon_p}{kT}\right] p^2\, dp$$

$$P = \frac{kT}{V}\exp\left[\frac{\mu}{kT}\right]\int_0^\infty \exp\left[-\frac{\epsilon_p}{kT}\right] g_s \frac{V}{h^3} 4\pi p^2\, dp \qquad (A)$$

hence $N = \int_0^\infty f(\epsilon_p)\, g(p)\, dp$

$$N = \exp\left[\frac{\mu}{kT}\right]\int_0^\infty \exp\left[-\frac{\epsilon_p}{kT}\right] g_s \frac{V}{h^3} 4\pi p^2\, dp \qquad (B)$$

and by comparision (A) and (B)

$$\boxed{p = \frac{N}{V} kT = n kT}$$ equation of state for a ideal classical gas

The average kinetic energy of a particle in a classical gas is equal to 3/2 kT if it is non relativistic and equal to 3kT if it is ultra-relativistic.

Chemical potential for a non-relativistic classical gas

$$N = A \exp\left(\frac{\mu}{kT}\right) \int_0^\infty \exp\left(-\frac{\epsilon_p}{kT}\right) p^2 \, dp \qquad \text{with} \quad A = 4\pi \frac{V}{h^3} g_s$$

$$\epsilon_p = mc^2 + \frac{p^2}{2m}$$

$$N = A \exp\left(\frac{\mu}{kT}\right) \int_0^\infty \underbrace{\exp\left(-\frac{mc^2}{kT} - \frac{p^2}{2mkT}\right)}_{\exp\left(-\frac{mc^2}{kT}\right) \cdot \exp\left(-\frac{p^2}{2mkT}\right)} p^2 \, dp$$

$$N = A \exp\left(\frac{\mu - mc^2}{kT}\right) \underbrace{\int_0^\infty \exp\left(-\frac{p^2}{2mkT}\right) p^2 \, dp}_{\text{substitution}} \qquad z = \frac{p^2}{2mkT}$$

$$\frac{dz}{dp} = \frac{p}{mkT} \qquad dp = \frac{mkT}{p} dz \Rightarrow \int_0^\infty \frac{e^{-z} \cdot 2mkT \cdot z \cdot mkT}{\sqrt{2mkTz}} dz$$

$$\underbrace{\frac{2m^2k^2T^2}{\sqrt{2mkT}}}_{\sqrt{2}(m \cdot kT)^{3/2}} \int_0^\infty \frac{z \, e^{-z}}{\sqrt{z}} dz =$$

$$\sqrt{2}(m \cdot kT)^{3/2} \underbrace{\int_0^\infty \exp(-z) \sqrt{z} \, dz}_{\sqrt{\pi}} = \frac{\sqrt{2\pi} (m \cdot kT)^{3/2}}{2}$$

$$N = 4\pi \frac{V}{h^3} g_s \exp\left(\frac{\mu - mc^2}{kT}\right) \frac{\sqrt{2\pi}\,(m\cdot kT)^{3/2}}{2}$$

$$= \frac{V}{h^3} g_s \exp\left(\frac{\mu - mc^2}{kT}\right) \underbrace{2\sqrt{2}\,(\pi m kT)^{3/2}}_{(2\pi m kT)^{3/2}}$$

$$\boxed{N = \frac{V}{h^3} g_s \exp\left(\frac{\mu - mc^2}{kT}\right) (2\pi m kT)^{3/2}}$$

rearranged

$$N = \exp\left(\frac{\mu - mc^2}{kT}\right) g_s \frac{V}{h^3} (2\pi m kT)^{3/2}$$

$$\frac{N}{V} = n = g_s \frac{(2\pi m\cdot kT)^{3/2}}{h^3} \exp\left(\frac{\mu - mc^2}{kT}\right)$$

$$\exp\left(\frac{\mu - mc^2}{kT}\right) = \frac{n h^3}{(2\pi m kT)^{3/2} g_s}$$

$$\mu - mc^2 = -kT \ln\left(\frac{(2\pi m kT)^{3/2} g_s}{n h^3}\right)$$

$$= -kT \ln\left(\frac{g_s}{n} \cdot \underbrace{\frac{(2\pi m kT)^{3/2}}{h^3}}_{\sqrt{\frac{(2\pi m kT)^3}{h^6}}}\right)$$

$$\sqrt{\frac{(2\pi m kT)^3}{h^6}} = \left(\frac{2\pi m kT}{h^2}\right)^{3/2}$$

„quantum concentration"

$$\boxed{n_Q = \left(\frac{2\pi m kT}{h^2}\right)^{3/2}}$$

Chemical potential for a ultra-relativistic classical gas

we neglect the rest energy mc^2: $\epsilon_p = pc$

$$N = A \exp\left(\frac{\mu}{kT}\right) \int_0^\infty \exp\left(-\frac{pc}{kT}\right) p^2 \, dp$$

Substitution: $z = \frac{pc}{kT}$ $\quad \frac{dz}{dp} = \frac{c}{kT}$ $\quad dp = \frac{kT}{c} dz$

$$p = \frac{z \cdot kT}{c}$$

$$\int_0^\infty \exp(-z) \frac{z^2 k^2 T^2 \cdot kT}{c^2 \cdot c} dz = \frac{k^3 T^3}{c^3} \underbrace{\int_0^\infty \exp(-z) \cdot z^2 \, dz}_{=2}$$

$$N = \frac{4\pi V}{h^3} g_s \cdot \frac{2 k^3 T^3}{c^3} \exp\left(\frac{\mu}{kT}\right)$$

$$\frac{N}{V} = n = \frac{8\pi \, g_s \, k^3 T^3}{h^3 c^3} \exp\left(\frac{\mu}{kT}\right)$$

$$\mu = -kT \ln\left[\frac{g_s}{n} \cdot \underbrace{8\pi \left(\frac{kT}{h \cdot c}\right)^3}_{\text{„quantum concentration"}}\right]$$

$$\boxed{n_Q = 8\pi \left[\frac{kT}{hc}\right]^3}$$

notation: n_{QNR} non-relativistic gas

n_{QUR} ultra-relativistic gas

2.2. Electrons in stars

If stellar matter compressed → electrons change their role

inside the sun form the electrons a classical gas → $n < n_Q$

first: dilute classical gas
second: $\underbrace{\text{dense quantum gas}}_{\text{would be relativistic}}$

→ compare electron particle density with quantum concentration!

$$n_Q \sim T^{3/2} \qquad n_Q \sim R^{-3/2} \qquad n \sim R^{-3}$$

⇒ The process of (core) contraction will lead to an electrons gas in which quantum effects are important

↳ if gas become more dense → e^- → relativistic

The degenerate electron gas

> Quantum effects dominate when the concentration of electrons becomes large compared with the quantum concentration n_Q

$n \gg n_Q$ is equivalent to $kT \ll \frac{h^2 n^{2/3}}{2\pi m_e}$

> A cold gas of electrons is called a degenerate gas because the electrons have fallen into quantum states with the lowest possible energy.

↳ Pauli exclusion principle

$f(\epsilon_p) = \left(\exp\left[(\epsilon_p - \mu)/kT\right] + 1\right)^{-1}$ fermi-dirac-distribution

set $\mu = \epsilon_F$ (≡ „fermi energy") for $T = 0\,K$

$$f(\epsilon_p) = \begin{cases} 1 & \epsilon_p \leq \epsilon_F \\ 0 & \epsilon_p > \epsilon_F \end{cases}$$

The energy of the most energetic electrons in a cold electron gas, ϵ_F, is called the fermi-energy (or, p_F, the fermi-momentum)

Important: The total number of electrons in a degenerate gas is the number of states with momentum less p_F (one electron pro state).

$f(\epsilon_p) = 1 \quad g_s = 2 \quad$ (2 spin states)

$$N = \int_0^{p_F} g_s \frac{V}{h^3} 4\pi p^2 \, dp = \frac{8\pi V}{3h^3} p_F^3$$

Fermi momentum

$$p_F = \left[\frac{3n}{8\pi}\right]^{1/3} \cdot h \qquad (\text{because } n = N/V)$$

The Broglie wavelength of the most energetic electrons in a degenerate gas ($\lambda = h/p_F$) is comparable with $n^{-1/3}$, the average distance between the electrons.

Equation of state of a degenerate e- gas

a) degenerate electron gas are non-relativistic

$p_F \ll mc \quad (\equiv n \ll (mc/h)^3 \quad \rightarrow h/mc = 2.4 \times 10^{-12}$ m

internal energy $\quad \epsilon_p = mc^2 + \frac{p^2}{2m}$

$$E = \int_0^{\infty} \epsilon_p \, f(\epsilon_p) \, g(p) \, dp$$

$$E = \int_0^{p_F} \epsilon_p \, g_s \frac{V}{h^3} 4\pi p^2 \, dp = N \left[mc^2 + \frac{3p_F^2}{10m}\right]$$

$$P = \frac{2}{3}\frac{N}{V} \cdot \frac{3p_F^2}{10m} \Rightarrow \boxed{P = n\frac{p_F^2}{5m}}$$

Pressure in an non-relativistic degenerate electron gas

$$P = \underbrace{\frac{h^2}{5m}\left[\frac{3}{8\pi}\right]^{2/3}}_{K_{NR}} n^{5/3} \qquad \boxed{P = K_{NR} \, n^{5/3}}$$

equation of state

b) ultra-relativistic degenerate electron gas

$n \gg n_{QUR}$ $\quad n \gg (mc/h)^3$ $\quad \epsilon_p = pc$

$$E = \int_0^{p_F} \epsilon_p g_s \frac{V}{h^3} 4\pi p^2 \, dp = N \frac{3}{4} p_F \cdot c$$

$$N = \int_0^{p_F} g_s \frac{V}{h^3} 4\pi p^2 \, dp = \frac{8\pi V}{3h^3} p_F^3$$

$$\boxed{E = \frac{3}{4} N p_F c}$$ internal Energy

$$P = \frac{N}{3V} \langle pc \rangle \Rightarrow P = n \frac{1}{4} p_F c$$

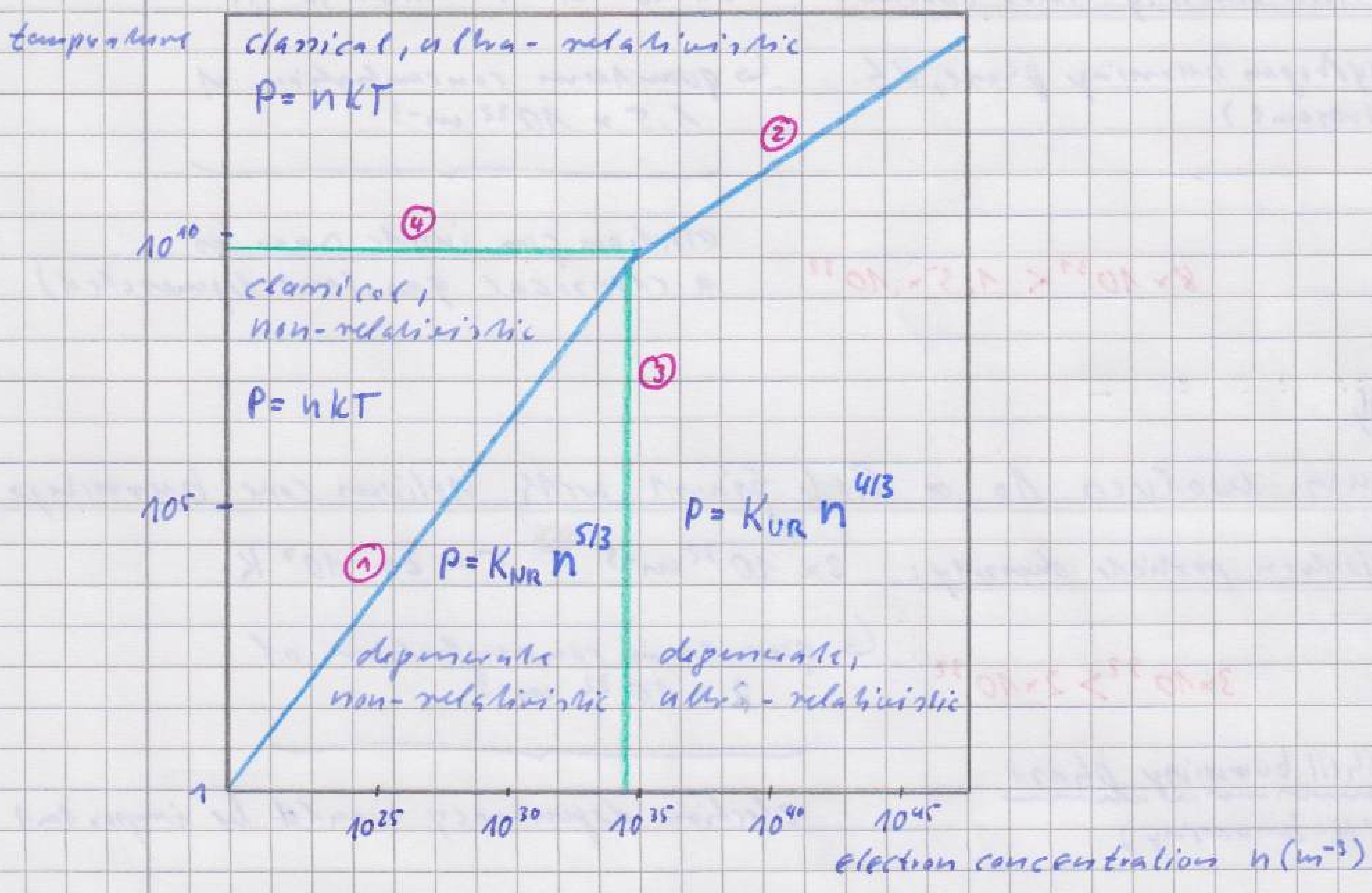
and this gives the equation of states:

$$P = \underbrace{\frac{hc}{4} \left[\frac{3}{8\pi}\right]^{1/3}}_{K_{UR}} n^{4/3} \qquad \boxed{P = K_{UR} \, n^{4/3}}$$

The pressure of a degenerate gas is an increasing function of the density, but the rate of increase becomes less rapid once the particles become ultra-relativistic.

↳ This is important for the stability of white dwarfs!

A density - temperature diagram



$n = n_{QNR} \approx 2\times 10^{21}\, T^{3/2}\ m^{-3}$ 1

$n = n_{QUR} \approx 8\times 10^{6}\, T^{3}\ m^{-3}$ 2

$n = (mc/h)^3 \approx 7\times 10^{34}\ m^{-3}$ 3

$T = mc^2/k \approx 6\times 10^{9}\ K$ 4

In practice the electron gas is not ideal because electrons interact.

↳ electrostatic corrections

⇒ As the density of a degenerate gas increases, electrostatic interactions become less important and the ideal gas approximation becomes more appropriate.

Electrons in the sun

electron density solar center: $8 \times 10^{31} m^{-3}$ $T \sim 1.6 \times 10^{7}$ K

(hydrogen burning phase, d.h. present)

↳ quantum concentration of $1.5 \times 10^{32} m^{-3}$

$8 \times 10^{31} < 1,5 \times 10^{32}$

electron gas inside sun is a classical gas (not degenerated)

⇓

sun wolves to a Red Giant with helium core burning

electron particle density: $3 \times 10^{32} m^{-3}$; $T \sim 2 \times 10^{7}$ K

↳ quantum concentration of $2 \times 10^{32} m^{-3}$

$3 \times 10^{32} > 2 \times 10^{32}$

⇒ Shell burning phase (H-burning)

electron degeneracy would be important

⇒ core burning phase (He-burning)

electron particle density: $3 \times 10^{34} m^{-3}$ $T \sim 10^{8}$ K

↳ quantum concentration of $2 \times 10^{33} m^{-3}$

$3 \times 10^{34} > 2 \times 10^{33}$

We have a cold, degenerate gas in which most of the electrons occupy the states of lowest energy

⇓

Even though 10^{8} K is cold enough for the electrons to be degenerate, it is hot enough to ignite the fusion of helium to form carbon.

If the core of the sun degenerated (→ Red Giant), then providing the electron gas the bulk of the pressure.

↳ the fusion control mechanism doesn't function

↳ The onset of helium burning in the sun will cause an explosive release of energy in a thermal runaway called a

Helium flash

Helium flash: the peak power could exceed the present luminosity of the sun by a factor of 10^{11}!

- but only a fraction of this energy will escape as radiation

↳ go into rapid expansion of the core which lifts the electron degeneracy.

And yet another interesting fact of the sun:

$\boxed{\text{The sun has insufficient mass to proceed beyond helium burning.}}$

↳ endpoint of sun's evolution is a white dwarf with a core composed mostly of carbon and oxygen. The mass is uncertain because of the uncertain mass losses during the red giant and planetary nebula phases of evolution.

Electrons in massive stars

Theoretical models indicate that a star with a mass greater than $11 M_\odot$ will normally evolve through all the stages of thermonuclear burning with no effects due to electron degeneracy.

↳ However, electron degeneracy plays a spectacular role at the end of the evolution of a very massive star.

- (eventually) $\underbrace{\text{forming a Fe-Core}}_{\text{core contracting} \rightarrow e^- \text{-degenerate}}$ ($T \approx 8 \times 10^9$ K, $\varrho \approx 4 \times 10^{12}$ kgm^{-3})

- if core mass $> \underbrace{1.4 M_\odot \text{ (Chandrasekhar mass)}}_{\text{rapidly collapses to a neutron star}}$

⇙ outer layers ejected by a supernova

⇘ neutrino-flash neutronization of the core's matter

$\boxed{\text{The collapse of the Fe-core is a direct result of the ultra-relativistic nature of the electron gas attempted to support it.}}$

2.3. Photons in stars

First approximation: a star consists of matter and radiation in thermodynamical equilibrium

but: the pressure due to radiation ("radiation pressure") inside a star can be nearly as important as the pressure due to electrons and ions.

The photon gas

Photon gas: an „ideal" gas of photons

↳ all "particles" move at the same speed, the „speed of light".

properties: the number of particles can change (photons as zero mass bosons can be created and destroyed)

Thermodynamic: $dE = TdS - PdV + \underbrace{\mu dN}_{\text{change of particle number}}$

At fixed energy E and Volume V, the number of photons changes until the entropy S is a maximum.

$$dE = TdS + \mu dN = 0 \implies \frac{\partial S}{\partial N} = -\frac{\mu}{T} = 0$$

free energy: $F = E - TS$ $\qquad dF = dE - d(TS) = dE - (TdS + SdT)$

$$dF = -SdT - PdV + \mu dN$$

A photon gas in equilibrium has zero chemical potential.

↳ we can set $\mu = 0$ in the Bose-Einstein distribution function.

$f(\epsilon_p) = (\exp(\epsilon_p)/kT) - 1)^{-1}$ → distribution function

$$\boxed{N(p)\,dp = \frac{1}{\exp(\epsilon_p/kT) - 1}\, g_s \frac{V}{h^3} 4\pi p^2\, dp}$$

$\epsilon_p = pc$

$g_s = 2$

Numbers of photons per unit volume

$$n = \frac{1}{V}\int_0^\infty N(p)\,dp = 8\pi\left[\frac{kT}{hc}\right]^3 \underbrace{\int_0^\infty \frac{x^2}{e^x-1}\,dx}_{\approx 2.4} \quad \text{with} \quad x = \frac{pc}{kT}$$

Energy per unit volume

$$u = \frac{1}{V}\int_0^\infty \underbrace{\epsilon_p}_{pc = x\,kT} N(p)\,dp = 8\pi\left[\frac{kT}{hc}\right]^3 \underbrace{\int_0^\infty \frac{x^3}{e^x-1}\,dx}_{\approx 6{,}49}$$

The exact solution of the integrals can be related to the Riemann zeta function ζ:

$$\int_0^\infty \frac{x^2}{e^x-1}\,dx = 2\,\zeta(3) = 2.404$$

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = 6\,\zeta(4) = \pi^4/15$$

Equations can be simplified:

$$n = bT^3 \qquad \text{where} \quad b = 2.404 \times \frac{8\pi k^3}{h^3c^3} = 2.03 \times 10^7\ K^{-3}\,m^{-3}$$

$$\boxed{u = aT^4} \qquad \text{where} \quad a = \frac{8\pi^5 k^4}{15h^3c^3} = 7.565 \times 10^{-16}\ J\,K^{-4}\,m^{-3}$$

Stefan-Boltzmann law

The average energy of a photon in a photon gas at temperature T is $2.70\,kT$.

<u>Summary:</u> non-relativistic $\frac{3}{2} kT$

ultra-relativistic $3 kT$

photon gas $2.70 kT$

Radiation pressure

$$P = \frac{N}{3V} \langle pc \rangle \Rightarrow P_r = \frac{1}{3} u = \frac{1}{3} a T^4$$

$$\boxed{P_r = \frac{1}{3} a T^4}$$

Planck's formula

$$N(p)\, dp = \left(\exp\left(\epsilon_p / kT\right) - 1\right)^{-1} g_s \frac{V}{h^3} 4\pi p^2 dp$$

$$\epsilon_p = pc = h\nu \qquad U = \frac{N \cdot h\nu}{V}$$

$$\underbrace{I_\nu\, d\nu}_{\text{intensity}} = \frac{c}{4} U_\nu\, d\nu = \frac{c}{4} \frac{h\nu}{\exp(h\nu/kT) - 1} \frac{8\pi\nu^2}{c^3} d\nu$$

$\hookrightarrow$ maximum: $\nu = 2.82\, kT/h$

$$\boxed{I_\nu\, d\nu = \frac{2\pi h \nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1} d\nu}$$

Radiation pressure in stars

Radiation pressure is significant in the very hot interiors of stars:

<u>Example:</u> Sun a) photosphere ($T \sim 6 \times 10^3$ K) $P_r = 0.33$ Pa

b) center ($T \sim 6 \times 10^6$ K) $P_r = 0.33 \times 10^{12}$ Pa

Radiation pressure can't be neglected in the interior of massive and very massive stars.

$$\frac{P_r}{P_g} \sim M^2$$

Thus the ratio of the radiation pressure generated by photons to the "gas" pressure generated by the electrons and ions increases with the mass of the star.

$\Rightarrow$ the "gas" pressure and the radiation pressure becomes comparable if the mass of the star exceeds $50 M_\odot$

$\hookrightarrow$ radiation pressure is likely to have a destabilizing effect on massive stars.

2.4. The Saha equation

Problem: Ionization of hydrogen

4 types of particles $\rightarrow$ equilibrium reactions

$$A + B \rightleftarrows C + D$$

if μ the chemical potential, then in chem. equilibrium:

$$\underbrace{\mu(A) + \mu(B) = \mu(C) + \mu(D)}_{\text{applied to the ionization of hydrogen}}$$

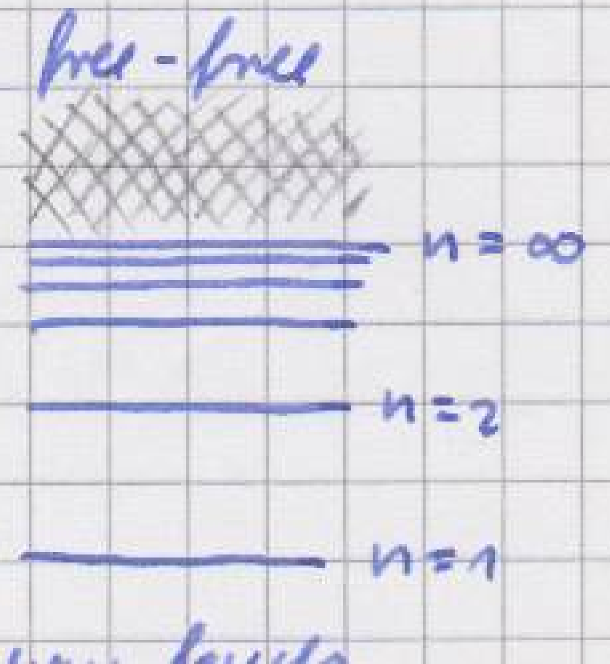
Energy levels of hydrogen

$\varepsilon_p = p^2/2m$ free electrons

$\varepsilon_\infty = 0$ serial limit for ionization

$\varepsilon_n = -13.6/n^2$ eV

$\varepsilon_1 = -13.6$ eV ground state



The interaction with photons can cause the hydrogen atom to be excited and ionized.

<u>Reaction:</u> $\gamma + \underset{\text{level-Number}}{H_n} \rightleftarrows \underbrace{e^- + p}_{\text{Ionization}}$

- the chemical potential of a Photon is zero!

<u>thermodynamical equilibrium:</u>

$$\mu(H_n) = \mu(e^-) + \mu(p)$$

We must calculating the chemical potentials:

$$\mu - mc^2 = -kT \ln\left[g_i n_Q / n\right] \qquad n_Q = \left[\frac{2\pi m k T}{h^2}\right]^{3/2}$$

a) electron: $\mu(e^-) = m_e c^2 - kT \ln\left[\dfrac{g_e n_{Qe}}{n_e}\right]$ $\quad g_e = 2$

b) proton: $\mu(p) = m_p c^2 - kT \ln\left[\dfrac{g_p n_{Qp}}{n_p}\right]$ $\quad g_p = 2$

c) neutral hydrogen atom:

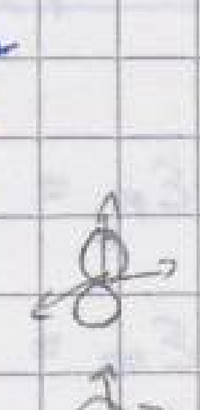
$$\mu(H_n) = \underbrace{m(H_n)c^2}_{m_e c^2 + m_p c^2 + E_n} - kT \ln\left[\frac{g(H_n)\, n_{Qp}}{\underbrace{n(H_n)}_{\text{concentration of hydrogen atoms in energetic state } n}}\right]$$

quantum concentration

> The number of hydrogen atom states $g(H_n)$ with energy E_n is determined by the degeneracy arising from the spin and the relative orbital angular momentum of the electron and proton in the atom.

→ <u>degeneracy:</u> there can be several orbital angular momentum states with the same energy E_n

↳ <u>example:</u> $n = 2$ 3 p-states, 1 s-state



$$\rightarrow g(H_n) = g_n \cdot g_e \cdot g_p \qquad \text{with } g_n = n^2$$

$$\mu(H_n) = \mu(e) + \mu(p)$$

$$\cancel{m_e c^2} + \cancel{m_p c^2} + \varepsilon_n - kT \ln \overbrace{\left[\frac{g(H_n)\, n_{Qp}}{n(H_n)}\right]}^{A}$$

$$= \cancel{m_e c^2} - kT \ln\left[\frac{g_e n_{Qe}}{n_e}\right] + \cancel{m_p c^2} - kT \ln\left[\frac{g_p n_{Qp}}{n_p}\right]$$

$$= -kT \ln \underbrace{\left[\frac{g_e g_p n_{Qe} n_{Qp}}{n_e n_p}\right]}_{B}$$

$$\varepsilon_n = kT \ln A - kT \ln B = kT \ln \frac{A}{B}$$

$$\varepsilon_n = -kT \ln \frac{A}{B} = -kT \ln \left[\frac{g_e g_p n_{Qe} \cancel{n_{Qp}}\, n(H_n)}{n_e n_p\, g(H_n)\, \cancel{n_{Qp}}}\right]$$

$$\exp\left(-\frac{\varepsilon_n}{kT}\right) = \frac{g_e g_p n_{Qe}\, n(H_n)}{n_e n_p\, g(H_n)} = \frac{n_{Qe}\, n(H_n)}{n_e n_p g_n}$$

$$(g(H_n) = g_n g_e g_p)$$

$$\boxed{\frac{n(H_n)}{n_e n_p} = \frac{g_n}{n_{Qe}} \exp\left(-\frac{\varepsilon_n}{kT}\right)}$$ Saha equation

$$n_{Qe} = \left[\frac{2\pi m_e kT}{h^2}\right]^{3/2} \approx 2.4146 \times 10^{21}\, T^{3/2}\, m^{-3}$$

The Saha-equation describe the result of a dynamic deadlock in which the reaction rate for

$$\gamma + H_n \rightarrow e^- + p$$

balances the rate for

$$e^- + p \rightarrow \gamma + H_n.$$

$$\Downarrow$$

$$\frac{n(H_n)}{n_e n_p} = \underbrace{f_n(T)}$$

to be proportional to the probability that an electron is bound, and inversely proportional to the probability that an electron is unbound.

Calculation of $f_n(T)$

probability that an electron is bound in a state with energy ε_n is proportional

$$g_e\, g_n \exp\left[-\frac{\varepsilon_n}{kT}\right]$$

$$\int_0^\infty \exp\left(-\frac{\varepsilon_p}{kT}\right) g_e \frac{4\pi}{h^3} p^2\, dp \qquad \varepsilon_p = \frac{p^2}{2m} \quad \text{(kinetic energy)}$$

$$z = \frac{p^2}{2mkT} \qquad p^2 = 2mkT\cdot z \qquad \frac{dz}{dp} = \frac{p}{mkT} \qquad dp = \frac{mkT}{p}\, dz$$

$$dp = \frac{m\cdot kT}{\sqrt{2mkTz}}\, dz = \frac{m\cdot kT\sqrt{2mkTz}}{2mkTz}\, dz = \left(\frac{2mkTz}{4z^2}\right)^{1/2} dz$$

$$= \left(\frac{mkT}{2z}\right)^{1/2} dz = \left(\frac{mkT}{2}\right)^{1/2} z^{-1/2}\, dz$$

$$\int_0^\infty \exp(-z)\, g_e \frac{4\pi}{h^3} 2mkT\cdot \underbrace{\frac{z}{\sqrt{z}}}_{=\sqrt{z}} \left(\frac{mkT}{2}\right)^{1/2} dz \qquad \left(\frac{4m^3k^3T^3}{2}\right)^{1/2} \Rightarrow (2mkT)^{3/2}$$

$$\frac{4\pi}{h^3} g_e (2mkT)^{3/2} \underbrace{\int_0^\infty \exp(-z)\sqrt{z}\, dz}_{\frac{\sqrt{\pi}}{2}} = \frac{4\pi \overbrace{\sqrt{\pi}}^{\pi^{3/2}} \cdot g_e (2mkT)^{3/2}}{2h^3}$$

$$= \frac{g_e (2\pi mkT)^{3/2}}{h^3} = g_e \sqrt{\frac{(2\pi mkT)^3}{\underbrace{h^6}_{(h^2)^3}}} = g_e \underbrace{\left(\frac{2\pi mkT}{h^2}\right)^{3/2}}_{n_{Q_e}}$$

$$\int_0^\infty \exp\left(-\underbrace{\frac{p^2}{2mkT}}\right) g_e \frac{4\pi}{h^3} p^2\, dp = g_e\, n_{Q_e}$$

probability e^--bound $= g_e g_n \exp\left[-\frac{\varepsilon_n}{kT}\right]$

probability e^--unbound $= g_e n_{Qe}$

$\Downarrow$

$$\frac{n(H_n)}{n_e n_p} = f_n(T) = \frac{g_e g_n \exp(-\varepsilon_n/kT)}{g_e n_{Qe}}$$ Saha-equation

Concentration of un-ionized hydrogen atoms

neutral part: $$\frac{n(H)}{n_e n_p} = \frac{1}{n_{Qe}} \sum_{n=1}^{n=\infty \text{ (strict limit)}} g_n \exp\left[-\frac{\varepsilon_n}{kT}\right]$$

$-\varepsilon_n = E_i + (\varepsilon_1 - \varepsilon_n)$

$$\frac{n(H)}{n_e n_p} = \frac{Z}{n_{Qe}} \exp\left[\frac{E_i}{kT}\right]$$ $\underbrace{E_i = -\varepsilon_1 = 13.6\,eV}_{\text{ionization energy}}$

partition function $$Z = \sum_{n=1}^{n=\infty} g_n \exp\left[-\frac{\varepsilon_n - \varepsilon_1}{kT}\right] \Rightarrow \infty$$

$(\varepsilon_n - \varepsilon_1)$ is the excitation energy of the n'th state.

Z is divergent. But in practice, Z is of the order of unity because the sum is terminated when the value of n corresponds to a state whose spatial extent is comparable with the distance between the gas particles.

→ replace n_p by $n(H^+)$

ionization part: $$\frac{n(H^+)}{n(H)} \approx \frac{n_{Qe}}{n_e} \exp\left[-\frac{E_i}{kT}\right]$$

↳ The ionization increases if the density of the gas decreases.

$$\ln\left[\frac{n(H^+)}{n(H)}\right] = F - \frac{E_i}{kT}$$

$$F = \ln\left[\frac{n_{Qe}}{n_e}\right]$$

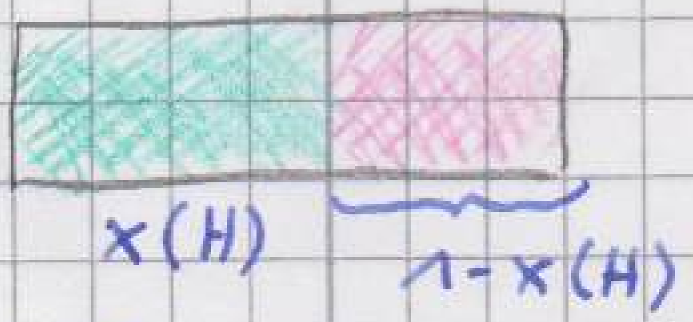
$\Rightarrow$ Thus when F is large, as in a very dilute electron gas, the onset of ionization occurs rapidly near $kT = E_i / F$.

2.5. Ionization in stars

→ star consists mainly of hydrogen

↳ concentration: $\varrho = [\underbrace{n(H)}_{\text{neutral}} + \underbrace{n(H^+)}_{\text{ionized}}]\, \overset{\text{mass of hydrogen atom}}{m_H}$

→ $x(H)$ is the ionized fraction of hydrogen



$$n_e = n(H^+) = \frac{x(H)\,\varrho}{m_H} \qquad n(H) = [1 - x(H)]\,\varrho / m_H$$

reverse Saha-equation

$$\frac{n(H)}{n(H^+)} = \frac{n_e}{n_{Qe}} \exp\left(\frac{E_i}{kT}\right) \qquad E_i = 13.6\ eV$$

$$\frac{[1 - x(H)]\,\varrho \cdot m_H}{m_H\, x(H) \cdot \varrho} = \frac{[1 - x(H)]}{x(H)} = \frac{x(H)\,\varrho}{m_H\, n_{Qe}} \exp\left(\frac{E_i}{kT}\right)$$

$$\frac{[1 - x(H)]}{(x(H))^2} = \frac{\varrho}{m_H \underbrace{n_{Qe}}_{2 \times 10^{21}\, T^{3/2}\, m^{-3}}} \exp\left(\frac{E_i}{kT}\right) \qquad \frac{E_i}{kT} \approx 1.578 \times 10^5\ K$$

$$\boxed{\frac{[1 - x(H)]}{x(H)^2} = \frac{\varrho}{m_H \cdot 2 \times 10^{21}\, T^{3/2}} \exp\left(\frac{1.578 \cdot 10^5}{T}\right)}$$

example: sun $\quad \varrho \approx 1.4 \times 10^3 \ kg \ m^{-3}$

$\bar{T} \approx 6 \times 10^6 K$

$$\frac{[1-x(H)]}{x(H)^2} \approx 0.03$$ 97 % of hydrogen is ionized

Complete ionization of a stellar gas

- mass fractions: X_1 hydrogen (1p)
  X_4 helium (2p, 2n)
  X_A „metals"

- neutral case: $n_1 = X_1 \varrho / m_H$
  $n_4 = X_4 \varrho / 4 m_H$
  $n_A = X_A \varrho / A m_H$

• neutral hydrogen atom $\xrightarrow{\text{ionized}}$ 1p + 1e⁻

• neutral helium atom $\xrightarrow{\text{ionized}}$ 1 nucleus + 2e⁻

• neutral atom with mass number A and atomic number Z

$\xrightarrow{\text{ionized}}$ 1 nucleus + Z e⁻ (A/2 particles)

The total number of particles per unit volume in a full ionized gas is:

$$n \approx 2n_1 + 3n_4 + \frac{A}{2} n_A = \left[2X_1 + \frac{3}{4} X_4 + \frac{1}{2} X_A\right] \frac{\varrho}{m_H}$$

Since $X_1 + X_4 + X_A = 1$, we have

$$\boxed{n \approx [1 + 3X_1 + 0.5 X_4] \frac{\varrho}{m_H}}$$

$$\bar{m} = \frac{\varrho}{n} \approx \frac{2m_H}{[1 + 3X_1 + 0.5X_4]}$$ average mass of the gas particles

example: Standard Solar model (Bahcall, 1989)

$X_1 = 0.71 \qquad X_4 = 0.27 \qquad X_A = 0.02$

The standard model predicts that hydrogen burning in the sun has reduced the hydrogen content and increased the helium content so that, at present, the mass fractions in the central regions are approximately

$$X_1 = 0{,}34 \qquad X_4 = 0{,}64 \qquad X_A = 0{,}02$$

$$\bar{m} = 0{,}85 \text{ amu}$$

Numbers of ions per unit volume in a fully ionized gas

$$n \approx 2n_1 + 3n_4 + \frac{A}{2} n_A$$

$$n_e \approx \left[1 + X_1\right] \frac{\varrho}{2m_H} \qquad \text{and} \qquad n_i \approx \left[2X_1 + 0{,}5\,X_4\right] \frac{\varrho}{2m_H}$$

$$(1 + 3X_1 + 0{,}5\,X_4) - (1 + X_1)$$

Stellar atmospheres

Spectral sequence: $\underbrace{O - B - A - F - G - K - M}$

reflects a steady decline in surface temperature

$30000\,K \longrightarrow 3000\,K$

Ionization energy

$\sim 5\,eV$	Li, Na, Mg, Al, K, Ca ...	predominantly } ionized
$10 - 20\,eV$	H, C, N, O, F, P, S, Cl, Ar	partially
$> 20\,eV$	He, Ne $\rightarrow$ only in the hottest stars	

Use of Saha-equation for Na, H and He

Na : 5.14 eV

H : 13.6 eV

He : 24.6 eV

$$\frac{n(Na^+)}{n(Na)} \approx \frac{10^{21}\, T^{3/2}}{n_e} \exp\Big[\underbrace{-60000/T}_{E_i/k_b}\Big]$$

$$\frac{n(H^+)}{n(H)} \approx \frac{10^{21}\, T^{3/2}}{n_e} \exp\left[-158000/T\right]$$

$$\frac{n(He^+)}{n(He)} \approx \frac{10^{21}\, T^{3/2}}{n_e} \exp\left[-286000/T\right]$$

Example: solar atmosphere

$$\frac{n(Na^+)}{n(Na)} \approx 10^{7}\, \frac{n(H^+)}{n(H)}$$ strong Na-absorption lines

$$\frac{n(He^+)}{n(He)} \approx 10^{-10}\, \frac{n(H^+)}{n(H)}$$ no/weak He-absorption lines

The ionization of metallic elements plays a crucial role in stellar atmospheres

↳ stellar matter consists largely of hydrogen and helium

↳ but the most of the free electrons are due to the ionization of metallic elements like sodium

$$\boxed{\text{The degree of ionization of other elements such as hydrogen and helium, depends on the electron concentration caused from metals.}}$$

Example: solar atmosphere

$$\underbrace{\frac{n(H^+)}{n(H)} \approx 10^{-4}}_{\text{partially ionized}} \qquad \frac{n(He^+)}{n(He)} \approx 10^{-14} \qquad \underbrace{\frac{n(Na^+)}{n(Na)} \approx 10^{3}}_{\text{predominantly ionized}}$$

The H^--Ion in stellar atmospheres

The H^--ion is a bound state of a proton and two electrons.

↳ two electron system like Helium

→ $Z = 1$; binding energy of second electron: $\underbrace{0.75\,eV}_{\sim 5\%\, E_i}$

$\lambda = hc/\vartheta \approx 1653\,nm$

photons with a wavelength as long as 1650 nm are absorbed and emitted by the reaction:

$$\gamma + H^- \rightleftarrows e^- + H$$

A transparent gas of hot, un-ionized hydrogen can be made opaque and luminous by the presence of free electrons.

↳ electrons comes from easily ionized metallic elements

M „metals"

electron concentration:

$$n_e = \underbrace{n(M^+)}_{\text{ions}} = x(M)\,\underbrace{[n(M) + n(M^+)]}_{\text{neutral part}}$$

application of the Saha-equation:

(assume, all metals have the same ionization energy as $\underbrace{\text{sodium}}_{E_i = -5.14\,eV}$)

$$\frac{\overbrace{[1 - x(M)]}^{\text{ionized part}}}{x(M)^2} \approx \frac{[n(M) + n(M^+)]}{10^{21}\,T^{3/2}} \exp[60000/T]$$

→ dynamic equilibrium concentration of H^-

$$\frac{n(H^-)}{n(H)} \approx \frac{n_e}{10^{21}\,T^{3/2}} \exp[8700/T]$$

→ couple between easily ionized metals and the concentration of loosely bound H^--ions in stellar atmospheres

↳ below 3000 K the gas is no longer luminous and opaque (visible radiation)

↳ important implication for the evolution of a star after it has left the main sequence

⇓

On leaving the main sequence, the luminosity increases and the surface temperature decreases, so the star moves upwards and to the right in the H-R diagram. However, the temperature of the visible surface cannot drop below 3000 K. Thus the star can only increase its luminosity by expanding at almost constant surface temperature. During this phase of evolution the star occupies an area of the H-R diagram called the giant branch („red giants", like Betelgeuse)

2.6. Reactions at high temperature

When the temperature becomes comparable with a billion degrees, the interaction of radiation with matter gives rise to two new processes:

- the production of electron-positron-pairs
- the photodisintegration of atomic nuclei

Electron-Positron pair production

$\Rightarrow kT \gtrsim m_e c^2$

$$\underbrace{\gamma + \gamma}_{\substack{\text{chemical} \\ \text{potential} = 0}} \rightleftarrows e^- + e^+ \Rightarrow \mu(e^-) + \mu(e^+) = 0$$

$$\Rightarrow n(e^-)\, n(e^+) = 4 n_0^2 \exp\left[-\frac{2 m_e c^2}{kT}\right]$$

(only valid for not degenerate e^--gas)

Special case: degenerate e⁻-gas

↳ pair production is only inhibited, if the electron can be filled a unoccupied quantum state

→ Thus pair production is favoured by high temperature and low densities.

Another important astrophysical implication of pair production is

$$\gamma + \gamma \rightleftarrows e^- + e^+ \rightarrow \underbrace{\nu_e + \overline{\nu_e}}_{\text{in } 1:10^{22} \text{ cases !}}$$

⇓

neutrino production in the hot central regions of highly evolved stars.

⇓

cooling effect

Energy loss by neutrinos can be important in stars if their core reaches a temperature of 10^9 K at a density where the electrons are not too degenerate.

⇒ neutrino cooling – leads to stimulate a faster rate of thermonuclear fusion in order to maintain steady conditions inside the stars

⇒ Energy loss by neutrinos accelerates the rate of evolution of the star.

Photodisintegration of nuclei

→ destroy atom nucleus → analogue ionisation by atoms

→ $T \gtrsim 3 \times 10^9$ K ; example: $\gamma + {}^{20}Ne \rightarrow {}^{16}O + {}^{4}He$

↳ $+ {}^{20}Ne \rightarrow {}^{24}Mg$

Photodisintegration plays the key role in silicon burning, the final stage of nuclear burning which leads to the formation of nuclei near Iron in the periodic table.

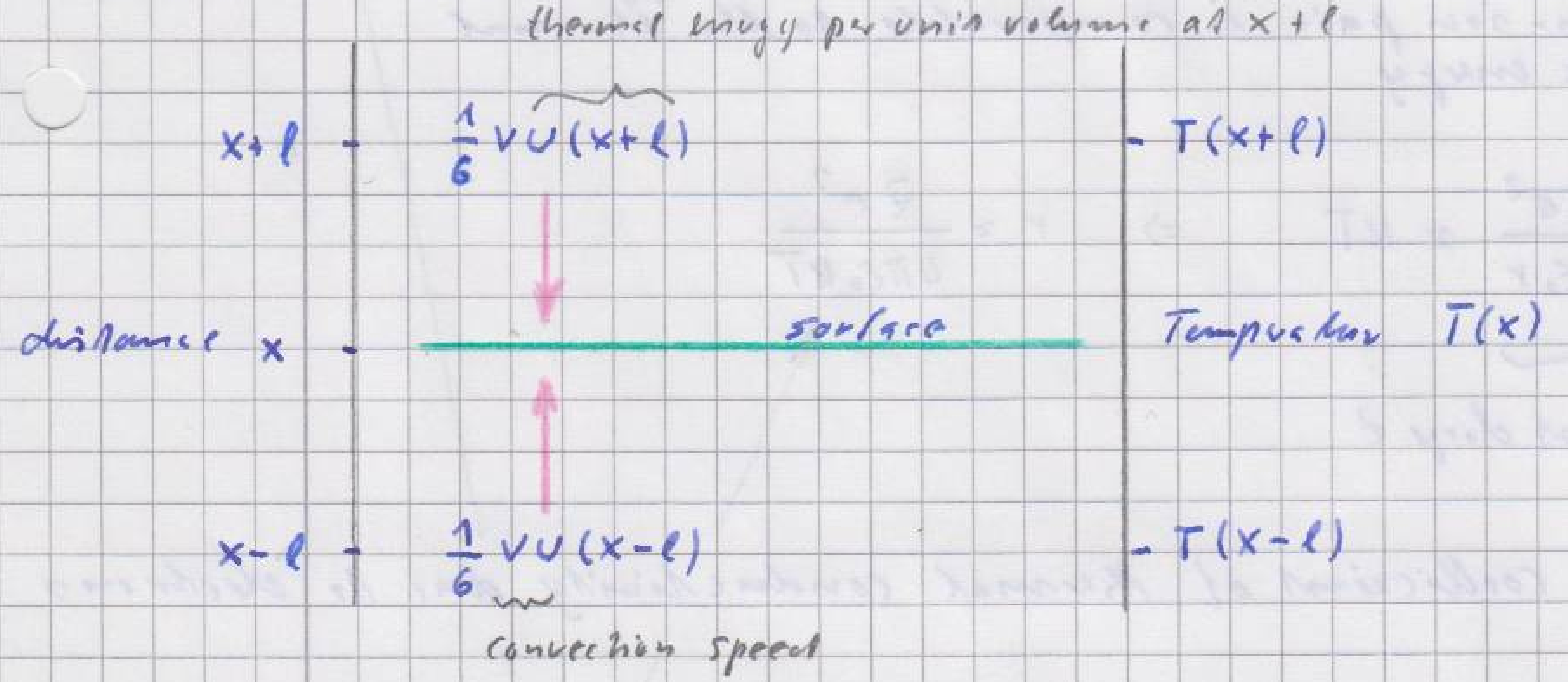
3. Heat transfer in stars

It gives two basic mechanisms for the transport of heat inside a star

- thermal conduction (→ radiative diffusion)
- convective heat transfer

⇒ Heat transfer is a complex and difficult subject

3.1. Heat transfer by random motion



heat stream: $j(x) \approx \frac{1}{6} v U(x-l) - \frac{1}{6} v U(x+l)$

$$\approx -\frac{1}{3} v l \frac{dU}{dx}$$

$$\underbrace{U = U(x) \quad T = T(x)}$$

$$\frac{dU}{dx} = \frac{dU}{dT}\frac{dT}{dx} = \underbrace{C}_{\text{heat capacity per unit volume}} \frac{dT}{dx}$$

$$\boxed{j(x) = -\underbrace{K}_{K \approx \frac{1}{3} v l C} \overbrace{\frac{dT}{dx}}^{\text{temperature gradient}}}$$

The flux density of heat across the surface at x

$K \approx \frac{1}{3} v l C$: coefficient of thermal conductivity of the gas

⇒ by photons: radiative diffusion

Random motion of electrons and ions

Classical electron gas with concentration n_e at temperature T:

$$U_e = \frac{3}{2} n_e k T \qquad C_e = \frac{3}{2} n_e k \qquad \bar{v}_e \approx \left(\frac{3kT}{m_e}\right)^{1/2}$$

→ heat transfer by e⁻-e⁻-collisions: not very effective

→ heat transfer by e⁻-Ion-collisions: more effective

σ electron-ion crosssection } can be estimated to πr^2

r is the distance at which the potential energy of an electron-ion pair is comparable to the thermal kinetic energy

$$\hookrightarrow \underbrace{\frac{Ze^2}{4\pi\varepsilon_0 r}}_{\text{Ion of charge } Z} \approx kT \quad \Rightarrow \quad r \approx \frac{Ze^2}{4\pi\varepsilon_0 kT}$$

Estimate: coefficient of thermal conductivity due to electrons

$$K_e \approx \frac{1}{3}\, \bar{v}_e \cdot \underbrace{\bar{l}}_{\text{mean free path}} \cdot C_e \qquad \bar{l} = \frac{1}{n_i \underbrace{\sigma}_{\text{cross section } = \pi r^2}}$$

r

$$K_e \approx \underbrace{\frac{1}{3} \cdot \frac{3}{2} n_e k}_{C_e} \cdot \underbrace{\frac{1}{n_i \pi r^2}}_{\bar{l}} \cdot \underbrace{\left[\frac{3kT}{m_e}\right]^{1/2}}_{\bar{v}_e}$$

$$\approx \frac{n_e k}{2\pi n_i} \left[\frac{3kT}{m_e}\right]^{1/2} \cdot \left.\frac{1}{r^2}\right\} = \left(\frac{4\pi\varepsilon_0 kT}{Ze^2}\right)^2$$

$$\boxed{K_e \approx \frac{k}{2\pi} \frac{n_e}{n_i} \left[\frac{3kT}{m_e}\right]^{1/2} \left[\frac{4\pi\varepsilon_0 kT}{Ze^2}\right]^2}$$

Estimate: coefficient of thermal conductivity due to ions

→ full ionized plasma: $n_e = Z n_i$

$$\boxed{K_i \approx \frac{1}{Z^2}\left[\frac{m_e}{m_i}\right]^{1/2} K_e}$$

⇒ $Z > 1$ and $m_i \gg m_e$ ⇒ $K_i \ll K_e$

ions: bad thermal conductors
electr.: good " " } by random motion

Thermal conductivity by electrons and ions is of minor importance in most stars

↳ exception: white dwarfs ⇒ $\underbrace{\text{degenerate electron gas}}_{\text{isotherm interior}}$

Random motion of photons

→ photons form a gas → photons move with speed of light

$$\underbrace{u_r = aT^4}_{\text{energy density}} \qquad \underbrace{C_r = 4aT^3}_{\text{thermal capacity}}$$

→ heat flux density due to radiative diffusion:

$$j(x) = -K_r \frac{dT}{dx} \qquad \underbrace{K_r}_{\text{coefficient of thermal conduction due to the random motion of photons}} \approx \frac{4}{3} c \bar{l} a T^3$$

What is the mean free path length of photons?

→ Thomson scattering by electrons

$$l = \frac{1}{\underbrace{n_e \sigma_T}_{\text{cross section for thomson scattering}}} \qquad \sigma_T = \frac{8\pi}{3}\left[\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right]^2$$

↳ - classical radiation of accelerating e^-
- quantum electrodynamics

thermal conduction due to radiative diffusion
thermal conduction due to random motion of e^-

↳
$$\frac{K_r}{K_e} \approx \sqrt{3}\, Z \underbrace{\frac{P_r}{P_e}}_{\text{pressure}} \left[\frac{m_e c^2}{kT}\right]^{5/2}$$

⇒ example: sun interior $T \approx 6 \times 10^6\,K$; $\rho \approx 1.4 \times 10^3\,kg\,m^{-3}$

$$\left.\begin{aligned} kT &\approx 10^{-3} m_e c^2 \\ P_r &\approx 3 \times 10^{11}\,Pa \\ P_e &\approx 7 \times 10^{13}\,Pa \end{aligned}\right\} K_r \approx 2 \times 10^5 K_e$$

⇓ conclusion

⇒ Radiative diffusion is a more effective mechanism for heat transfer in the sun than thermal conduction by electrons.

Absorption problem:

impuls and energy conservation → photon cannot be absorbed by an interaction with a free particle

⇓

presence of ions

⇒ bound-free absorption (photoionization)

free-free absorption (inverse bremsstrahlung)

lead to a mean free path which varies with the frequency of the photon

↳ absorption coefficient

Black body radiation

$$I_\nu \, d\nu = \frac{c}{4} U_\nu \, d\nu = \frac{c}{4} \frac{h\nu}{\exp(h\nu/kT) - 1} \frac{8\pi \nu^2}{c^3} d\nu$$

$$U_\nu \, d\nu = \frac{h\nu}{\exp(h\nu/kT) - 1} \, 8\pi \frac{\nu^2}{c^3} \, d\nu$$

$$C_\nu \, d\nu = \frac{\partial U_\nu}{\partial T} \, d\nu$$

If $\bar{l}_\nu$ is the mean free path length at frequency ν, the coefficient of conduction due to photons at all frequencies is

$$K_r = \int_0^\infty \frac{1}{3} c \bar{l}_\nu C_\nu \, d\nu$$

$$\Rightarrow \quad \bar{l} = \frac{\int_0^\infty \bar{l}_\nu C_\nu \, d\nu}{4aT^3} \quad \Big\} \text{ „Rosseland average"}$$

$\Rightarrow$ is likely to be dominated by contributions at frequencies near $2.8\,kT/h$, where C_ν is a maximum, and at frequencies where $\bar{l}_\nu$ is large

↳ i.e. where the stellar material is almost transparent

mean free path length

n_e electron concentration
n_i ion "

σ_e, σ_i interaction cross sections

$$\bar{l} = \frac{1}{n_e \sigma_e + n_i \sigma_i}$$

$n_e, n_i \sim \varrho$

$$= \frac{1}{\underbrace{\varrho \, \kappa}_{\text{opazität}}}$$

Hence $j(x) = -K_\nu \frac{dT}{dx}$ $\quad K_\nu \approx \frac{4}{3} c \bar{l} a T^3$

$\Rightarrow$ radiative diffusion

$$\boxed{j(x) = -\frac{4ac}{3} \frac{T^3}{\varrho \kappa} \frac{dT}{dx}}$$ flux density

- Opacity is a very important property of stellar material

Summary

Bound-free absorption is important at low temperature where a large fraction of the atoms are only partially ionized. Free-free absorption dominates at high temperatures where ionization nears completion.

$\hookrightarrow$ These mechanisms give a frequency averaged opacity which increases with density and decreases with temperature roughly in accordance with

$$\boxed{\kappa \sim \varrho T^{-3.5}}$$ Kramers law

Influence of electron scattering (high temperature, low density)

$$\kappa_{es} = n_e \sigma_T / \varrho = (1 + X_1) \sigma_T / 2 m_H$$

$$\approx \underbrace{(1 + X_1)}_{\text{mass fraction of hydrogen}} \times 0.02 \ m^2 \ kg^{-1}$$

Example: sun / photons

	κ	ϱ	$\bar{l}$
0.0 $R_\odot$	0.1 $m^2\,kg^{-1}$	1.5×10^5 $kg\,m^{-3}$	0.07 mm
0.6 $R_\odot$	1.0	3.5×10^2	3.0
0.9 $R_\odot$	10	1.2×10	8.0

3.2. Heat transfer by convection

→ collective motion of particles ("pockets of gas")

Gravity provides this force field in a star.

- A rising pocket of stellar gas may sometimes find itself in a cooler and more dense environment, and it will continue to rise because of its buoyancy.
- A falling pocket of gas will continue to fall if it finds itself in a warmer, less dense environment.

↳ lead to heat transfer by convection

⇒ Convection is so efficient that it will dominate other heat transfer mechanisms.

> Convection only takes place if the magnitude of the temperature gradient exceeds a certain critical value.

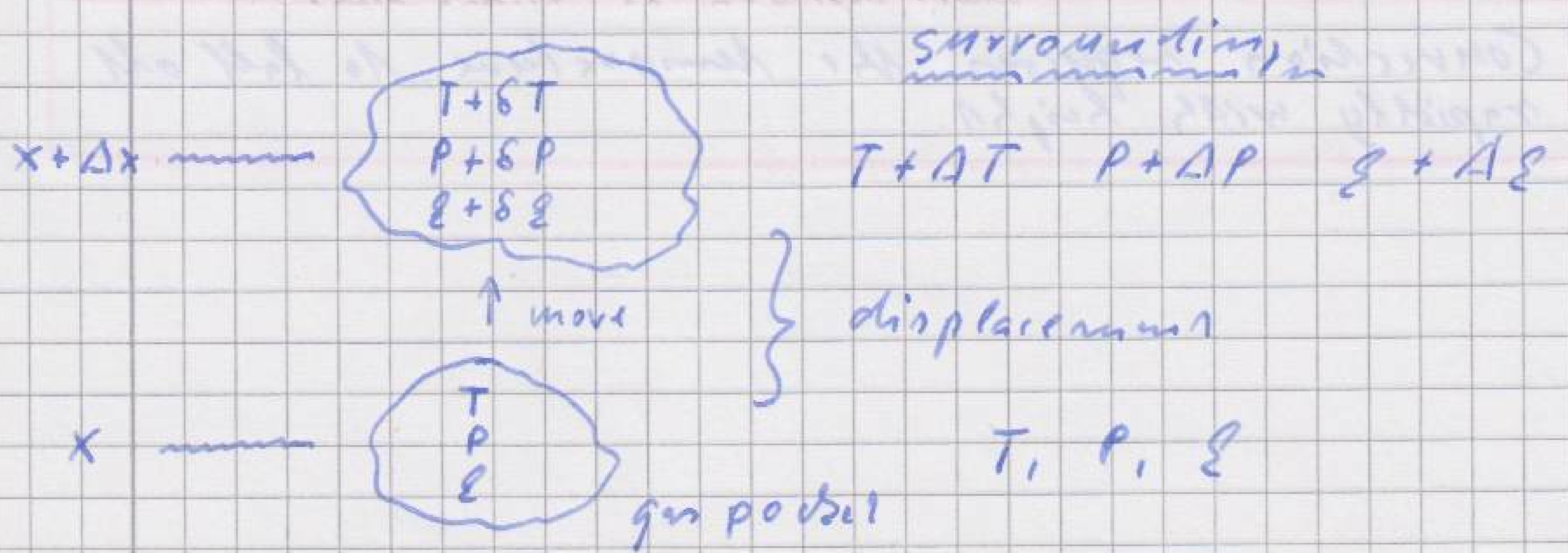
Critical conditions for convection

→ ideal gas in a gravitational field

height x $\quad T, P, \varrho$

height $x+\Delta x$ $\quad T+\Delta T,\ P+\Delta P,\ \varrho+\Delta\varrho$

$$\varrho \sim P/T \quad \Rightarrow \quad \frac{\Delta\varrho}{\varrho} = \frac{\Delta P}{P} - \frac{\Delta T}{T} \qquad \text{(ideal gas law)}$$



assume: displacement in insufficient time for heat conduction or heat exchange with environment

↳ gas expanded adiabatically

→ adiabatic process: $P \sim \varrho^{\gamma}$

$$\boxed{\frac{\delta\varrho}{\varrho} = \frac{1}{\gamma}\frac{\delta P}{P}}$$ ⇒ gas pocket will be buoyant ⇑

If the pocket contains gas which is less dense than the surrounding gas

↳ convection is possible if

$$\delta\varrho < \Delta\varrho \iff \frac{1}{\gamma}\frac{\delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T}$$

⇓

We set $\delta P = \Delta P$

$$\frac{\Delta T}{T} < \frac{(\gamma - 1)}{\gamma}\frac{\Delta P}{P}$$

This gives the critical temperature gradient for convection

$$\boxed{\frac{\Delta T}{\Delta x} \approx \frac{dT}{dx} < \frac{(\gamma-1)}{\gamma}\frac{T}{P}\frac{dP}{dx}}$$

Schwarzschild criteria

(temperature and pressure gradients are both negative!)

⇒ Convection requires the temperature to fall off rapidly with height.

From statistical mechanics: adiabatic index of an ideal classical gas is related to the number of classical degrees of freedom of the gas particles

$$\gamma = \frac{C_p}{C_v} = \frac{1 + \frac{s}{2}}{\frac{s}{2}}$$

↳ each with a average thermal energy of $\frac{1}{2} kT$

- ideal gas particles: (pointlike)

$\underbrace{3 \text{ Translation}, 0 \text{ rotation}}_{s=3}$

$\Downarrow$

$\gamma = 5/3$

$\Downarrow$

absorb heat

$\Downarrow$ (not pointlike)

$\underbrace{\text{rotational degrees of freedom} \quad \text{Vibrational " " "}}$ } γ would smaller

- ionization of atoms
- dissociation of molecules

Another important factor:

$$\frac{dT}{dx} < \frac{(\gamma - 1)}{\gamma} \frac{T}{P} \frac{dP}{dx} \Rightarrow \underbrace{\frac{dP}{dx} = -g\,\varrho(x)}$$

In regions where g is small the pressure falls off gradually and convection is more easily induced

In practice, convection currents transfer heat very effectively. Indeed, the process is so efficient that, in many circumstances, all the heat generated can be transported as soon as the temperature gradient reaches the critical value.

3.3. Temperature gradients in Stars

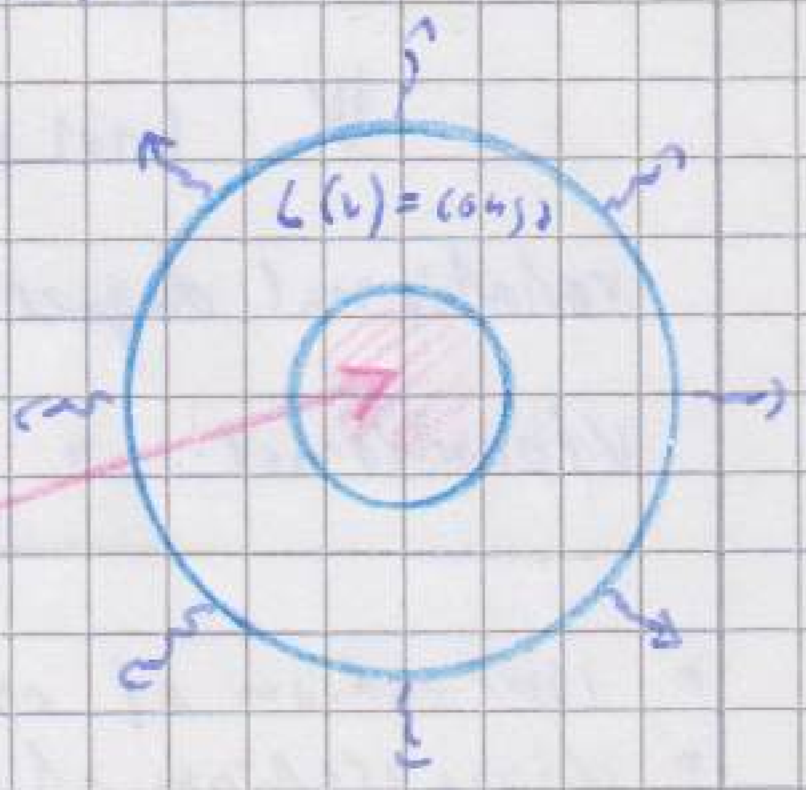
⇒ The temperature gradient at a point inside a star is determined by the rate of flow of energy towards the surface and the mechanism governing this energy flow.

$L(r)$ energy flow rate through a spherical surface of radius r within a star

$\varepsilon(r)$ nuclear power generated per unit volume at r

↳ power in a shell $(r, r+dr)$

$$\Rightarrow \varepsilon(r) \cdot 4\pi r^2 \, dr$$



This gives the outward power flow:

$$\frac{dL}{dr} = 4\pi r^2 \, \varepsilon(r)$$

Outside any central generating regions, $L(r)$ becomes constant and approaches the surface luminosity of the star.

radiative diffusion as the dominant heat transfer mechanism

$$j(x) = -\frac{4ac}{3}\frac{T^3}{\varrho k}\frac{dT}{dx} \quad \text{flux density of radiation}$$

$\hat{=} \ \varepsilon(r>x) \Rightarrow L(r) = 4\pi r^2 j(r)$ (outside energy release region)

$$\frac{L(r)}{4\pi r^2} = -\frac{4ac}{3\varrho(r)} \underbrace{\frac{T(r)^3}{k(r)}}_{\text{opacity}} \cdot \underbrace{\frac{dT}{dr}}_{\text{temperature gradient}}$$

$$\boxed{\left[\frac{dT}{dr}\right]_{rad} = - \frac{3\varrho(r)\kappa(r)}{4ac\,T(r)^3} \cdot \frac{L(r)}{4\pi r^2}}$$

Temperature gradient for radiation diffusion

<u>Example:</u> sun $\quad L = 4 \times 10^{26}\,W$ (surface, $r > 0.4\,R_\odot$)

$r = 0.4\,R_\odot \rightarrow T \approx 5 \times 10^6\,K$; $\varrho \approx 5 \times 10^3\,kg$; $\kappa \approx 0.5\,m^2\,kg^{-1}$

$\hookrightarrow \quad \frac{dT}{dr} \approx -0.03\,K\,m^{-1}$

$\Rightarrow$ the solar interior is dense and opaque and radiation can indeed diffuse slowly to and from regions which are in local thermodynamic equilibrium (LTE)

$$\boxed{\left[\frac{dT}{dr}\right]_{conv} = \frac{(\gamma - 1)}{\gamma}\,\frac{T}{P}\,\underbrace{\frac{dP}{dr}}}$$

Temperature gradient for heat convection

(critical value)

$$\frac{dP}{dr} = -\frac{G\,m(r)\,\varrho(r)}{r^2}$$

in hydrostatic equilibrium

In practice, convection dominates radiative diffusion whenever the temperature gradient reaches the critical value

$\Rightarrow$ Convection is particularly important in

- ionization zones (star atmospheres)
- cores of massive main sequence stars

<u>Example:</u> convection zone in the sun

from a depth of $0.287 \pm 0.003\,R_\odot \rightarrow \underbrace{\text{Photosphere}}_{\text{granules}}$

Convection in the Centre of stars

Convection can also be important in the central energy-generating regions of stars. The most favoured situation occurs when thermonuclear power is generated in a small region near the centre. In this case, large amounts of energy flow through a region where the acceleration due to gravity is low: the pressure falls off gradually, and a rising pocket of gas is more likely to remain buoyant because it need not expand much.

Model: $\frac{L(r)}{m(r)} \rightarrow$ becomes critical in star centre

$$\left[\frac{dT}{dr}\right]_{rad} = \left[\frac{dT}{dr}\right]_{conv}$$

$$\frac{3\varrho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2} = \frac{(\gamma-1)}{\gamma}\frac{T}{P}\frac{Gm(r)\varrho}{r^2}$$

$\hookrightarrow$ replace $\frac{aT^4}{3} \Rightarrow P_r$ (radiation pressure)

$$\boxed{\left[\frac{L(r)}{m(r)}\right]_{crit} = \frac{(\gamma-1)}{\gamma}\frac{16\pi Gc}{\kappa}\frac{P_r}{P}}$$

Convection sets on, when $L(r)/m(r) < \left[L(r)/m(r)\right]_{crit}$

> Convection occurs in the cores of massive main sequence stars, where hydrogen burning takes place by the carbon-nitrogen cycle.

$\hookrightarrow$ power generation $\sim T^{17} \Rightarrow$ in centre a small region which convection dominated

3.4. Cooling of white dwarfs

A white dwarf is a inert star with no internal power source.

Interior: dense gas of classical ions and a degenerate electron gas

Surface: thin envelope of a classical gas

A white dwarf cools predominantly by the conduction of heat by electrons in the interior, and by the diffusion of radiation through the outer envelope.

$\Rightarrow$ The cooling time is long because of the high thermal energy of the ions in the interior and the high opacity of the gas composing the envelope.

$\Downarrow$

timescale ~ Billion years

A simple cooling model

Assumes:
- metal like core with uniform temperature
- insulating layer of ionized gas

Temperature of the isotherm core: T_I

↳ thermal energy per ion: $\frac{3}{2} k T_I$

$\Downarrow$

heat transfer: radiative diffusion across the envelope

The insulating properties of the outer envelope control the energy loss to outer space and thereby determine the relation between the luminosity L of the star and the steadily declining internal temperature T_I.

How change the pressure, temperature and density in the outer envelope of the white dwarf!

→ Surface layer: ideal classical gas $P = \rho k T / \bar{m}$

→ pressure gradient: $\frac{dP}{dr} = -\frac{GM\rho(r)}{r^2}$

→ temperature gradient: $\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)}{4ac[T(r)]^3}\cdot\frac{L}{4\pi r^2}$

⇒ in this layer is no energy generation!

combine pressure and temperature gradient

⇓

$$\frac{dP}{dT} = \left[\frac{16\pi a c G M}{3L}\right]\cdot\frac{T^3}{\underbrace{\kappa}_{\text{opacity}}}$$

The opacity depends on the temperature, density and chemical composition

model:
- 90% mass H + He
- 10% heavy elements
- ↳ opacity is caused by bound-free-absorption

$$\kappa = \kappa_0 \rho T^{-3.5} = 4.34\times 10^{19}\rho T^{-3.5}\ \mathrm{m^2\,kg^{-1}}$$

and with use of the ideal gas equation:

$$\kappa = \left[\frac{\kappa_0 \bar{m}}{k}\right] P T^{-4.5}$$

$$\boxed{\frac{dP}{dT} = \underbrace{\left[\frac{16\pi a c G k M}{3\kappa_0 \bar{m} L}\right]}_{C}\cdot\frac{T^{7.5}}{P}}$$

for the white dwarfs envelope